Stoch. Proc. HW assignment 8. Due Friday, November 3 at start of class

1. Consider a time-homogeneous Poisson point process with rate λ on $\mathbb{R}_+$. $T(n)$ denotes the time of the n'th arrival, N_t denotes the number of arrivals up to time t and $N_{(s,t]}$ denotes $N_t - N_s$, i.e., the number of arrivals between s and t, where $0 \le s \le t$.

   (a) Find $\mathbb{P}[\, N_4 = k \,|\, N_3 = n \,]$ for any $k, n \in \mathbb{N}$ and $\mathbb{E}[\, N_4 \,|\, N_3 = n \,]$, $\mathrm{Var}[\, N_4 \,|\, N_3 = n \,]$ for any $n \in \mathbb{N}$.

   (b) Find $\mathbb{P}[\, N_{(3,4]} = k \,|\, N_5 = n \,]$ for any $k, n \in \mathbb{N}$ and $\mathbb{E}[\, N_{(3,4]} \,|\, N_5 = n \,]$, $\mathrm{Var}[\, N_{(3,4]} \,|\, N_5 = n \,]$, $n \in \mathbb{N}$.

   (c) Find $\mathbb{P}[\, N_{(3,6]} = k \,|\, N_4 = n \,]$ for any $k, n \in \mathbb{N}$ and $\mathbb{E}[\, N_{(3,6]} \,|\, N_4 = n \,]$, $\mathrm{Var}[\, N_{(3,6]} \,|\, N_4 = n \,]$, $n \in \mathbb{N}$.

Solution:

(a) $N_4 = N_3 + N_{(3,4]}$, where $N_3 \sim \mathrm{POI}(3\lambda)$ and $N_{(3,4]} \sim \mathrm{POI}(\lambda \cdot (4-3))$ are independent. Thus if we condition on $N_3 = n$ then the conditional distribution of N_4 is the same as the distribution of $n + N_{(3,4]}$, where $N_{(3,4]} \sim \mathrm{POI}(\lambda)$, since conditioning on the value of the random variable N_3, which is independent of $N_{(3,4]}$, will not affect the distribution of $N_{(3,4]}$. Thus

$$\mathbb{P}[\, N_4 = k \,|\, N_3 = n \,] = \mathbb{P}[\, n + N_{(3,4]} = k \,] = \mathbb{P}[\, N_{(3,4]} = k - n \,] = \begin{cases} 0 & \text{if } n > k \\ e^{-\lambda} \frac{\lambda^{k-n}}{(k-n)!} & \text{if } n \le k \end{cases}$$

$\mathbb{E}[\, N_4 \,|\, N_3 = n \,] = \mathbb{E}[n + N_{(3,4]}] = n + \mathbb{E}[N_{(3,4]}] = n + \lambda$

$\mathrm{Var}[\, N_4 \,|\, N_3 = n \,] = \mathrm{Var}[n + N_{(3,4]}] = \mathrm{Var}[N_{(3,4]}] = \lambda$

(b) $N_5 = N_{(0,3]} + N_{(3,4]} + N_{(4,5]}$, where $N_{(0,3]} \sim \mathrm{POI}(3\lambda)$, $N_{(3,4]} \sim \mathrm{POI}(\lambda)$, $N_{(4,5]} \sim \mathrm{POI}(\lambda)$ are independent. Let $X = N_{(0,3]} + N_{(4,5]}$, thus $N_5 = X + N_{(3,4]}$ and $X \sim \mathrm{POI}(4\lambda)$ (by the merging property of Poisson random variables, see page 108), moreover X and $N_{(4,5]}$ are independent.

$$\mathbb{P}[\, N_{(3,4]} = k \,|\, N_5 = n \,] = \mathbb{P}[\, N_{(3,4]} = k \,|\, X + N_{(3,4]} = n \,] = \frac{\mathbb{P}[\, N_{(3,4]} = k \,,\, X + N_{(3,4]} = n \,]}{\mathbb{P}[\, X + N_{(3,4]} = n \,]} =$$

$$\frac{\mathbb{P}[\, N_{(3,4]} = k \,,\, X = n - k \,]}{\mathbb{P}[\, N_5 = n \,]} \overset{(*)}{=} \frac{e^{-\lambda} \frac{\lambda^k}{k!} \cdot e^{-4\lambda} \frac{(4\lambda)^{n-k}}{(n-k)!}}{e^{-5\lambda} \frac{(5\lambda)^n}{n!}} = \binom{n}{k} \left(\frac{1}{5}\right)^k \left(\frac{4}{5}\right)^{n-k},$$

where $(*)$ holds since $N_{(3,4]} \sim \mathrm{POI}(\lambda)$ and $X \sim \mathrm{POI}(4\lambda)$, and they are independent.

Alternative solution: we know that given that five arrivals occurred on the interval $[0,5]$, the locations of these points $U_1, \dots, U_5$ are i.i.d. with uniform distribution on the interval $[0,5]$ (see page 120). Thus, if we condition on $N_5 = n$ then $N_{(3,4]} = \sum_{k=1}^n \mathbb{1}[3 < U_k \le 4]$ is the number of points (out of n) that fall in the interval $(3,4]$. Now $\mathbb{P}[3 < U_k \le 4] = \frac{4-3}{5} = \frac{1}{5}$, thus if we condition on $N_5 = n$ then the conditional distribution $N_{(3,4]}$ is $\mathrm{BIN}(n, \frac{1}{5})$, therefore $\mathbb{P}[\, N_{(3,4]} = k \,|\, N_5 = n \,] = \binom{n}{k} \left(\frac{1}{5}\right)^k \left(\frac{4}{5}\right)^{n-k}$.

$\mathbb{E}[\, N_{(3,4]} \,|\, N_5 = n \,] = n \cdot \frac{1}{5}$, this is the expected value of $\mathrm{BIN}(n, \frac{1}{5})$.

$\mathrm{Var}[\, N_{(3,4]} \,|\, N_5 = n \,] = n \cdot \frac{1}{5} \cdot \frac{4}{5}$, this is the variance of $\mathrm{BIN}(n, \frac{1}{5})$.

(c) $N_{(3,6]} = N_{(3,4]} + N_{(4,6]}$, where $N_{(4,6]} \sim \mathrm{POI}(2\lambda)$, and $N_{(4,6]}$ is independent of anything that happens up to time 4, in particular $N_{(3,4]}$ and $N_{(4,6]}$ are conditionally independent given $N_4 = n$. If we condition on $N_4 = n$ then the conditional distribution of $N_{(3,4]}$ is $\mathrm{BIN}(n, \frac{1}{4})$, since a point which is uniformly distributed on $[0,4]$ will fall in $(3,4]$ with probability $1/4$. Therefore

$$\mathbb{P}[\, N_{(3,6]} = k \,|\, N_4 = n \,] = \sum_{\ell=0}^{k} \mathbb{P}[\, N_{(3,4]} = \ell,\, N_{(4,6]} = k - \ell \,|\, N_4 = n \,] \overset{(*)}{=}$$

$$\sum_{\ell=0}^{k} \mathbb{P}[\, N_{(3,4]} = \ell \,|\, N_4 = n \,] \mathbb{P}[\, N_{(4,6]} = k - \ell \,] = \sum_{\ell=0}^{k} \binom{n}{\ell} \left(\frac{1}{4}\right)^\ell \left(\frac{3}{4}\right)^{n-\ell} e^{-2\lambda} \frac{(2\lambda)^{k-\ell}}{(k-\ell)!},$$

where in $(*)$ we used that $N_{(3,4]}$ and $N_{(4,6]}$ are conditionally independent given $N_4 = n$ and that $N_{(4,6]}$ is independent of N_4.

$\mathbb{E}[\, N_{(3,6]} \,|\, N_4 = n \,] = \mathbb{E}[\, N_{(3,4]} \,|\, N_4 = n \,] + \mathbb{E}[\, N_{(4,6]} \,] = n/4 + 2\lambda$, since the expectation of $\mathrm{BIN}(n, \frac{1}{4})$ is $n/4$ and the expectation of $\mathrm{POI}(2\lambda)$ is 2λ.

$\mathrm{Var}[\, N_{(3,6]} \,|\, N_4 = n \,] \overset{(*)}{=} \mathrm{Var}[\, N_{(3,4]} \,|\, N_4 = n \,] + \mathrm{Var}[\, N_{(4,6]} \,] = n\frac{3}{16} + 2\lambda$, where $(*)$ holds since $N_{(3,4]}$ and $N_{(4,6]}$ are conditionally independent given $N_4 = n$, moreover the variance of $\mathrm{BIN}(n, \frac{1}{4})$ is $n\frac{1}{4}\frac{3}{4}$ and the variance of $\mathrm{POI}(2\lambda)$ is 2λ.

2. The lifetime of a light bulb has $\Gamma[2,1]$ distribution, in other words the density function of the lifetime of a light bulb is $te^{-t}\mathbb{1}[t \geq 0]$. If a light bulb burns out, I immediately replace it with a new one. At time zero, I start with a new light bulb.

   (a) Find the density function of the time when the third light bulb burns out.

   (b) Find the probability that at time $t = 5$ the third light bulb is on.

   (c) Denote by β_t the remaining lifetime of the light bulb that is on at time t. In other words, β_t is the length of the time interval that starts with t and ends with the next light bulb-switch. Find $\mathbb{P}(\beta_t \geq s)$ for any $s, t \in \mathbb{R}_+$. Find the density function $f_t(s)$ of the random variable β_t for any $t \in \mathbb{R}_+$. Find $\lim_{t\to\infty} f_t(s)$.
   Hint: The questions (a),(b),(c) become much easier if you find the PPP hidden in the exercise!

Solution: A random variable with $\Gamma[2,1]$ distribution is a sum of two independent random variables with EXP(1) distribution. Therefore, in order to obtain the renewal times of a renewal process where the renewal intervals have i.i.d. $\Gamma[2,1]$ distribution, it is enough to take a Poisson point process with rate 1 and erase the first, third, fifth, etc. points.

Let $\tau_1, \tau_2, \ldots$ denote i.i.d. random variables with EXP(1) distribution. Let $\widetilde{\tau}_k = \tau_{2k-1} + \tau_{2k}$ for $k = 1, 2, 3, \ldots$. Let $T(n) = \tau_1 + \cdots + \tau_n$ and $\widetilde{T}(n) = \widetilde{\tau}_1 + \cdots + \widetilde{\tau}_n = \tau_1 + \cdots + \tau_{2n} = T(2n)$. Then $\widetilde{T}(n)$ is the time when the n'th light bulb burns out. Denote by $N_t = \max\{n : T(n) \leq t\}$ and $\widetilde{N}_t = \max\{n : \widetilde{T}(n) \leq t\} = \max\{n : T(2n) \leq t\}$, thus $\widetilde{N}_t$ is the number of light bulb changes up to time t.

(a) $\widetilde{T}(3) = T(6) \sim \Gamma(6,1)$, thus the density function is $f(t) = e^{-t}\frac{t^5}{5!}\mathbb{1}[t \geq 0]$.

(b) $\{\widetilde{T}(2) \leq 5\} \setminus \{\widetilde{T}(3) \leq 5\} = \{T(4) \leq 5\} \setminus \{T(6) \leq 5\} = \{4 \leq N_5 < 6\} = e^{-5}\frac{5^4}{4!} + e^{-5}\frac{5^5}{5!}$

(c) $\{\beta_t \geq s\}$ occurs if and only if there is no light bulb-switch in the interval $[t, t+s]$, i.e., either N_t is even and $N_{[t,t+s)} \leq 1$ or N_t is odd and $N_{[t,t+s)} = 0$. Thus

$$\mathbb{P}(\beta_t \geq s) = \mathbb{P}[\, N_t \text{ is even and } N_{[t,t+s)} \leq 1\,] + \mathbb{P}[\, N_t \text{ is odd and } N_{[t,t+s)} = 0\,] \stackrel{(*)}{=}$$
$$\mathbb{P}[\, N_t \text{ is even}\,]\mathbb{P}[\, N_{[t,t+s)} \leq 1\,] + (1 - \mathbb{P}[\, N_t \text{ is even}\,])\mathbb{P}[\, N_{[t,t+s)} = 0\,] \stackrel{(**)}{=}$$
$$\frac{1+e^{-2t}}{2}e^{-s}(1+s) + \frac{1-e^{-2t}}{2}e^{-s} = e^{-s} + \frac{1+e^{-2t}}{2}e^{-s}s$$

where in$(*)$ we used that N_t and $N_{[t,t+s)}$ are independent in a PPP, and in $(**)$ we used that

$$\mathbb{P}[\, N_t \text{ is even}\,] = \sum_{k=0}^{\infty} \mathbb{P}[N_t = 2k] = \sum_{k=0}^{\infty} e^{-t}\frac{t^{2k}}{(2k)!} = e^{-t}\sum_{k=0}^{\infty}\frac{t^{2k}}{(2k)!} = e^{-t}\cosh(t) = \frac{1+e^{-2t}}{2}.$$

We obtain the density function

$$f_t(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\mathbb{P}(\beta_t \geq s) = e^{-s} + \frac{1+e^{-2t}}{2}(e^{-s}s - e^{-s}) = \frac{1-e^{-2t}}{2}e^{-s} + \frac{1+e^{-2t}}{2}e^{-s}s.$$

We obtain the limit $f_\infty(s) = \lim_{t\to\infty} f_t(s) = \frac{1}{2}e^{-s}(1+s)$.

Remark: Quite similarly to the solution of HW6.1(b), one obtains that for any renewal process with absolutely continuous renewal times, the stationary probability density function of the remaining lifetime is

$$f_\infty(s) = \frac{\mathbb{P}[\widetilde{\tau}_1 \geq s]}{\mathbb{E}[\widetilde{\tau}_1]},$$

where $\widetilde{\tau}_1$ is the first renewal interval. In the case when $\widetilde{\tau}_1 \sim \Gamma[2,1]$, we have $\mathbb{E}[\widetilde{\tau}_1] = 2$ and $\mathbb{P}[\widetilde{\tau}_1 \geq s] = \mathbb{P}[\widetilde{N}_s = 0] = \mathbb{P}[N_s \leq 1] = e^{-s}(1+s)$.

3. It rains one hundred times a year on average. Let us assume that the storms are instantaneous and that they arrive according to a PPP. An old gardener waters his garden if it has not been watered (by either rain or himself) in the last 48 hours.

   (a) What is the distribution of the number of storms between two manual waterings?

   (b) What is the expected time that elapses between two manual waterings?

   (c) Roughly how many times does he have to water his garden manually this year?

Solution:

(a) The intensity of the rain PPP per day is $\lambda = \frac{100}{365}$. The length of a dry time interval is $t_0 = 2$ days. The inter-arrival time intervals between rains are $\tau_1, \tau_2, \ldots$ i.i.d. with $\text{EXP}(\lambda)$ distribution. If we denote $p := \mathbb{P}(\tau_n > t_0) = e^{-\lambda t_0}$ and we denote by X the the number of storms between two manual waterings, then $X \sim \text{GEO}(p)$ (pessimistic geo. distribution), i.e.,

$$\mathbb{P}(X = k) = p(1-p)^k = e^{-\lambda t_0}(1 - e^{-\lambda t_0})^k = e^{-\frac{200}{365}}(1 - e^{-\frac{200}{365}})^k, \qquad k = 0, 1, 2, \ldots$$

(b) Assuming that there was a manual watering at time zero, denote by Y the time that elapses until the next manual watering. We want to find $m := \mathbb{E}(Y)$. First let us find $\mathbb{E}(\, \tau_1 \,|\, \tau_1 \leq t_0 \,)$. Here is how to calculate this without integration:

$$\frac{1}{\lambda} = \mathbb{E}(\, \tau_1 \,) = \mathbb{E}(\, \tau_1 \,|\, \tau_1 \leq t_0 \,)\mathbb{P}(\tau_1 \leq t_1) + \mathbb{E}(\, \tau_1 \,|\, \tau_1 > t_0 \,)\mathbb{P}(\tau_1 > t_1) =$$
$$\mathbb{E}(\, \tau_1 \,|\, \tau_1 \leq t_0 \,)(1 - e^{-\lambda t_0}) + \mathbb{E}(\, \tau_1 \,|\, \tau_1 > t_0 \,)e^{-\lambda t_0} \overset{(*)}{=}$$
$$\mathbb{E}(\, \tau_1 \,|\, \tau_1 \leq t_0 \,)(1 - e^{-\lambda t_0}) + (t_0 + \frac{1}{\lambda})e^{-\lambda t_0},$$

where $(*)$ holds since $\mathbb{E}(\, \tau_1 \,|\, \tau_1 > t_0 \,) = t_0 + \lambda$ by the *memoryless property* of exponential distribution: given $\tau_1 > t_0$, the time that we have to wait until the first rain after t_0 has exponential distribution with parameter λ. Rearranging the formula $\frac{1}{\lambda} = \mathbb{E}(\, \tau_1 \,|\, \tau_1 \leq t_0 \,)(1 - e^{-\lambda t_0}) + (t_0 + \frac{1}{\lambda})e^{-\lambda t_0}$ we obtain

$$\mathbb{E}(\, \tau_1 \,|\, \tau_1 \leq t_0 \,) = \frac{1}{\lambda} - t_0 \frac{e^{-\lambda t_0}}{1 - e^{-\lambda t_0}}. \tag{1}$$

Now we can calculate

$$m = \mathbb{E}(Y) = \mathbb{E}(\, Y \,|\, \tau_1 \leq t_0 \,)\mathbb{P}(\tau_1 \leq t_1) + \mathbb{E}(\, Y \,|\, \tau_1 > t_0 \,)\mathbb{P}(\tau_1 > t_1) \overset{(**)}{=}$$
$$\mathbb{E}(\, \tau_1 + Y^* \,|\, \tau_1 \leq t_0 \,)(1 - e^{-\lambda t_0}) + t_0 e^{-\lambda t_0} = (\mathbb{E}(\, \tau_1 \,|\, \tau_1 \leq t_0 \,) + \mathbb{E}(\, Y^* \,|\, \tau_1 \leq t_0 \,))(1 - e^{-\lambda t_0}) + t_0 e^{-\lambda t_0} \overset{(***)}{=}$$
$$(\frac{1}{\lambda} - t_0 \frac{e^{-\lambda t_0}}{1 - e^{-\lambda t_0}} + m)(1 - e^{-\lambda t_0}) + t_0 e^{-\lambda t_0} = \left(\frac{1}{\lambda} + m\right)(1 - e^{-\lambda t_0}),$$

where in $(**)$ we denoted by Y^* the time that elapses between τ_1 and the first manual watering after τ_1 and in $(***)$ we used (1) and that Y^* is independent of τ_1 and $\mathbb{E}(Y^*) = m$.

Rearranging this we obtain that the the expected number of days that elapses between two manual waterings is

$$m = \frac{1}{\lambda} \frac{1 - e^{-\lambda t_0}}{e^{-\lambda t_0}} = \frac{e^{\lambda t_0} - 1}{\lambda} = \frac{e^{\frac{200}{365}} - 1}{\frac{100}{365}} \approx 2.66$$

(c) Roughly $365/m \approx 137.2$ waterings per year by the law of large numbers for renewal processes.