Stoch. Proc. HW assignment 9. Due Friday, November 17 at start of class

Note: Each of the 3 questions is worth 10 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises with a horizontal line. Highlight the final answer. Paper format is preferred, but if you submit your homework electronically, pdf format is preferred.

1. *On Murphy's law: why does my tram have so many passengers?* Trams arrive according to a PPP, once in five minutes on average. Passengers arrive at my station according to a PPP, one passenger per second on average. The trams have already been in service for a long time, all day, every day.

   (a) I arrive at the station, I let the first tram go away, and then I take the next tram. However, all of the other passengers take the first tram that they see. What is the distribution of the number of other passengers that I take the tram with? What is the expectation of the number of other passengers that I take the tram with?

   (b) I arrive at the station (independently from other passengers) and I take the first tram. What is the distribution of the number of other passengers that I take the tram with (i.e., what is the probability of the event that I take the tram with k other passengers)? What is the expectation of the number of other passengers that I take the tram with?

2. *A population model with immigration and killing (but no reproduction).* Let us consider the Markov chain (X_n) with state space $S = \{0, 1, 2, \dots\}$ and transition matrix

$$p(x, y) = \sum_{z=0}^{y} \binom{x}{z} p^z (1-p)^{x-z} e^{-\lambda} \frac{\lambda^{y-z}}{(y-z)!}, \qquad x, y \in S,$$

   where $p \in (0, 1)$ and $\lambda \in (0, +\infty)$. X_n denotes the number of individuals in a population in round n.

   (a) Explain how to obtain X_{n+1} from X_n in plain words using the notion of „killing" and „immigration".
   Hint: Read the official solution of HW8.1 carefully.

   (b) Find the entries of the n-step transition matrix $p^{(n)}(x, y)$, $x, y \in S$.
   Hint: Assuming that $X_0 = x$, first try to express the distribution of X_n by iterating the „killing" and „immigration" operations. Then you will realize that $p^{(n)}(x, y)$ has a form similar to $p(x, y)$.

   (c) Find the stationary distribution of (X_n).

3. Let T denote a random variable with distribution $\mathbb{P}(T = t_k) = p_k$, where $t_1, t_2, \dots \in \mathbb{R}_+$ and $\sum_{k=1}^{\infty} p_k = 1$. Starting at time zero, satellites are launched at times of a PPP with rate λ. A satellite stops working after a random amount of time. The lifetimes of satellites are independent from each other and their launching times. The lifetime of a satellite has the same distribution as T. Let X_t denote the number of working satellites at time $t \in \mathbb{R}_+$.

   (a) Find the distribution of X_t.

   (b) Find the limiting distribution of X_t as $t \to \infty$.