Stoch. Proc. HW assignment 9. Due Friday, November 17 at start of class

1. *On Murphy's law: why does my tram have so many passengers?* Trams arrive according to a PPP, once in five minutes on average. Passengers arrive at my station according to a PPP, one passenger per second on average. The trams have already been in service for a long time, all day, every day.

   (a) I arrive at the station, I let the first tram go away, and then I take the next tram. However, all of the other passengers take the first tram that they see. What is the distribution of the number of other passengers that I take the tram with? What is the expectation of the number of other passengers that I take the tram with?

   (b) I arrive at the station (independently from other passengers) and I take the first tram. What is the distribution of the number of other passengers that I take the tram with (i.e., what is the probability of the event that I take the tram with k other passengers)? What is the expectation of the number of other passengers that I take the tram with?

Solution:

(a) When the first tram goes away, I am the only passenger at the tram station. The expected arrival time of the second tram is 300 seconds, thus the PPP of tram arrivals is of rate $1/300$. Let's merge this with the PPP of passengers, which is of rate 1. The merged PPP is of rate $301/300$. We can obtain the original tram and passenger Poisson point processes from the merged process by colouring each arrival as a „tram" with probability $1/301$ and a „passenger" with probability $300/301$. Thus the number X of passengers before the first tram has pessimistic GEO(1/301) distribution: $\mathbb{P}(X = k) = \frac{1}{301}\left(\frac{300}{301}\right)^k$, $k = 0, 1, 2, \ldots$, and $\mathbb{E}(X) = 300$.

(b) Let us say that I arrive at time zero. The past and the future tram arrivals are independent Poisson point processes. The past and the future passenger arrivals are independent Poisson point processes. Thus if we denote by X_1 the number of passengers that were already there waiting for a tram when I arrived, and by X_2 the number of passengers that arrive between my arrival and the next tram's arrival, then X_1 and X_2 are independent with pessimistic GEO(1/301) distribution. The number Y of other passengers that I take the tram with is $Y = X_1 + X_2$. We have already calculated the distribution of Y in class, see page 97-98 of the scanned lecture notes (negative binomial distribution):

$$\mathbb{P}(Y = k) = \binom{k+1}{k} p^2 (1-p)^k = (k+1)\left(\frac{1}{301}\right)^2 \left(\frac{300}{301}\right)^k, \qquad k = 0, 1, 2, \ldots$$

$\mathbb{E}(Y) = \mathbb{E}(X_1 + X_2) = \mathbb{E}(X_1) + \mathbb{E}(X_2) = 300 + 300 = 600.$

Remark: The difference between the solution of (a) and (b) is another incarnation of the *waiting time paradox* (already discussed that the Remark after the solution of HW6.1). Another way to arrive at the result of part (b) given the result of part (a) is as follows: if we arrange passengers into groups and the cardinality of the groups are i.i.d. with pessimistic GEO(1/301) distribution (i.e., the probability that the group is of size k is $p_k = \frac{1}{301}\left(\frac{300}{301}\right)^k$, where $k = 0, 1, 2, 3, \ldots$), and then we choose a passenger (me) uniformly from all of the passengers who took the tram in 2017 in this tram stop, then the distribution of the size of my group will be *size-biased*, i.e., the probability that my group is of size k will be proportional to kp_k. Since $\sum_{k=0}^{\infty} kp_k = 300$, the probability that my group (including me) is of size k is $q_k := \frac{1}{300}kp_k$, where $k \geq 1$. The probability that I take the tram with k other passengers is q_{k+1}, where $k \geq 0$. Thus if we denote by Y the number of other passengers that I take the tram with then for any $k = 0, 1, 2, \ldots$ we have

$$\mathbb{P}(Y = k) = q_{k+1} = \frac{1}{300}(k+1)p_{k+1} = \frac{1}{300}(k+1)\frac{1}{301}\left(\frac{300}{301}\right)^{k+1} = (k+1)\left(\frac{1}{301}\right)^2\left(\frac{300}{301}\right)^k.$$

This is exactly the same formula as the one derived in part (b).

2. *A population model with immigration and killing (but no reproduction).* Let us consider the Markov chain (X_n) with state space $S = \{0, 1, 2, \dots\}$ and transition matrix

$$p(x,y) = \sum_{z=0}^{y} \binom{x}{z} p^z (1-p)^{x-z} e^{-\lambda} \frac{\lambda^{y-z}}{(y-z)!}, \qquad x, y \in S,$$

where $p \in (0,1)$ and $\lambda \in (0, +\infty)$. X_n denotes the number of individuals in a population in round n.

(a) Explain how to obtain X_{n+1} from X_n in plain words using the notion of „killing" and „immigration".

(b) Find the entries of the n-step transition matrix $p^{(n)}(x,y)$, $x, y \in S$.

(c) Find the stationary distribution of (X_n).

Solution:

(a) The number of individuals in our population in round n is X_n. Let us assume that $X_n = x$. In the next round, every individual dies with probability $1-p$, independently of other individuals. Denote by Z_{n+1} the number of individuals that did not die. Thus $Z_{n+1} \sim \mathrm{BIN}(x,p)$. Also, W_{n+1} new individuals immigrate, where $W_{n+1} \sim \mathrm{POI}(\lambda)$. Thus $X_{n+1} = Z_{n+1} + W_{n+1}$. Now for any $x, y \in S$

$$\mathbb{P}(\, X_{n+1} = y \,|\, X_n = x \,) = \sum_{z=0}^{y} \mathbb{P}[Z_{n+1} = z]\mathbb{P}[W_{n+1} = y - z] = \sum_{z=0}^{y} \binom{x}{z} p^z (1-p)^{x-z} e^{-\lambda} \frac{\lambda^{y-z}}{(y-z)!}$$

(b) We will show that if $X_0 = x$ then X_n will be the sum of a $\mathrm{BIN}(x, p_n)$ random variable (the individuals that are still there from the start and did not die yet) and an independent $\mathrm{POI}(\lambda_n)$ random variable (the immigrants).

Indeed: $p_0 = 1$ and $\lambda_0 = 0$.

At the end of first round $p_1 = p$ and $\lambda_1 = \lambda$, this is just the probabilistic meaning of the one-step transition rule that we discussed in the solution of part (a).

We will argue that $p_{n+1} = p \cdot p_n$ and $\lambda_{n+1} = p\lambda_n + \lambda$. Indeed:

- If we consider the x individuals that were there at the start, each of them is still alive after n rounds with probability p_n, now in round $n+1$, each of the remaining guys stays alive with probability p, thus by the end of round $n+1$ each of them will be alive with probability $p_{n+1} = p_n \cdot p$, independently form the others.
- Now if the number of immigrants at the end of round n has $\mathrm{POI}(\lambda_n)$ distribution and each of them stays alive with probability p (independently from the others) then (by the colouring property) the number of guys that are still alive has $\mathrm{POI}(p\lambda_n)$ distribution. Also, $\mathrm{POI}(\lambda)$ new immigrants arrive and merge with those that are already there, thus the number of immigrants at the end of round $n+1$ will have $\mathrm{POI}(p\lambda_n + \lambda)$ distribution.

Now it is easy to see that $p_n = p^n$ and $\lambda_n = \lambda \sum_{k=0}^{n-1} p^k = \lambda \frac{1-p^n}{1-p}$ for $n = 1, 2, 3, \dots$, and thus

$$p^{(n)}(x,y) = \sum_{z=0}^{y} \binom{x}{z} p_n^z (1-p_n)^{x-z} e^{-\lambda_n} \frac{\lambda_n^{y-z}}{(y-z)!}, \qquad x, y \in S$$

(c) $\lim_{n\to\infty} p_n = 0$ and $\lim_{n\to\infty} \lambda_n = \frac{\lambda}{1-p} =: \lambda_\infty$, thus

$$\lim_{n\to\infty} p^{(n)}(x,y) = e^{-\lambda_\infty} \frac{\lambda_\infty^y}{y!}, \qquad x, y \in S.$$

In words: ultimately each individual of the original population dies and the distribution of immigrants converges to $\mathrm{POI}(\lambda_\infty)$. Thus the unique stationary distribution is $\pi(x) = e^{-\lambda_\infty} \frac{\lambda_\infty^x}{x!}, x \in S$. Also note that $\lambda_\infty = p\lambda_\infty + \lambda$, i.e., λ_∞ is the fixed point of the iteration $\lambda_{n+1} = p\lambda_n + \lambda$.

Remark: This model is a discrete time version of the M/M/∞ queue, see page page 159-160 of the scanned lecture notes.

3. Let T denote a random variable with distribution $\mathbb{P}(T = t_k) = p_k$, where $t_1, t_2, \dots \in \mathbb{R}_+$ and $\sum_{k=1}^{\infty} p_k = 1$. Starting at time zero, satellites are launched at times of a PPP with rate λ. A satellite stops working after a random amount of time. The lifetimes of satellites are independent from each other and their launching times. The lifetime of a satellite has the same distribution as T. Let X_t denote the number of working satellites at time $t \in \mathbb{R}_+$.

   (a) Find the distribution of X_t.

   (b) Find the limiting distribution of X_t as $t \to \infty$.

Solution:

(a) Let us color the satellites by their lifetime: if it is t_i then we give the satellite color i. By the colouring property of PPP's, the PPP which consists of the launching times of the satellites with lifetime t_i is a time-homogeneous PPP with intensity $p_i\lambda$, moreover these PPP's are independent.

At time t a satellite of color i is alive if its launching occurred after $t - t_i$. Thus, the number of satellites of color i at time t has $\mathrm{POI}(p_i\lambda \cdot \min\{t_i, t\})$ distribution, since the time interval that starts at $\max\{0, t - t_i\}$ and ends at t has length $\min\{t_i, t\}$. Merging gives that the total number of working satellites at time t has Poisson distribution with parameter $\sum_{i=1}^{\infty} p_i\lambda \cdot \min\{t_i, t\} = \lambda\mathbb{E}(\min\{T, t\})$

(b) $\lim_{t\to\infty} \mathbb{E}(\min\{T, t\}) = \mathbb{E}(T)$ by the monotone convergence theorem, thus the limiting distribution of X_t as $t \to \infty$ is $\mathrm{POI}(\lambda\mathbb{E}(T))$.

Remark: If the lifetime of a satellite is exponentially distributed with rate μ then (X_t) is in fact an $\mathrm{M/M/\infty}$ queue, see page 159-160 of the scanned lecture notes.