Midterm Exam - September 29, 2017, Stochastic Processes

Family name ______________________ **Given name** ______________________

Signature ______________________ **Neptun Code** ______________________

No calculators or electronic devices are allowed. One formula sheet with 15 formulas is allowed.

1. University students sit in a row on a test exam. The leftmost student knows the correct answer to a yes/no question. The others have no clue, so they try to copy the solution of the student sitting on their left. With probability 3/4, each student succeeds in copying the answer written by the student on his left, but with probability 1/4 they write exactly the opposite answer as the student on their left. First the leftmost student writes her answer down, then the second student, then the third one, etc.

   (a) Can you model this process with a Markov chain? Write down the state space and the transition matrix.

   (b) What is the probability that the third student gives the correct answer?

   (c) What is the probability that the tenth student gives the correct answer?

   Solution:

   (a) Let X_n be equal to 1 is the n'th student writes the correct answer and let X_n be equal to 2 if the n'th student writes the wrong answer. Then (X_n) is a Markov chain with state space $S = \{1, 2\}$, initial state $X_1 = 1$ and transition matrix $\underline{\underline{P}} = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix}$.

   (b) $\mathbb{P}[X_3 = 1] = \mathbb{P}[X_3 = 1, X_2 = 1] + \mathbb{P}[X_3 = 1, X_2 = 2] = \frac{3}{4}\frac{3}{4} + \frac{1}{4}\frac{1}{4} = \frac{10}{16} = \frac{5}{8}$.

   (c) Let $\mu_n = \mathbb{P}[X_n = 1]$. Then $\mu_1 = 1$ and $\mu_{n+1} = \frac{3}{4}\mu_n + \frac{1}{4}(1 - \mu_n) = \frac{1}{2}\mu_n + \frac{1}{4}$. The fixed point of this recursion is $\mu = \frac{1}{2}$. The general solution of the corresponding homogeneous recursion is $\mu_n = C2^{-n}$ thus the general solution of the original inhomogeneous recursion is $\mu_n = \frac{1}{2} + C2^{-n}$. If we want $\mu_1 = 1$ then we need to chose $C = 1$ and thus $\mu_n = \frac{1}{2} + 2^{-n}$. In particular, the tenth student gives the correct answer with probability $\mu_{10} = \frac{1}{2} + 2^{-10}$ (still a bit better than guessing the answer!)

2. There are three chairs in a barbershop, including the one that the barber uses to cut the hair of clients. The other two chairs are used by clients waiting for their hair to be cut. While he cuts the hair of a client, three things can happen: zero, one or two new clients can possibly arrive, with probabilities 1/2, 1/3 and 1/6, respectively. If there is no vacant chair then possible new clients walk away immediately without sitting down.

   (a) Let X_n denote the number of clients waiting when the n'th haircut is finished. Can you model this process with a Markov chain? Write down the state space and the transition matrix.

   (b) The barber just finished cutting the hair of a client and there are two further clients waiting for a haircut. It takes twenty minutes for the barber to cut the hair of a client. What is the expected time (in minutes) until the barber can take a break?

   Solution:

   (a) The number X_n can be zero, one or two. Thus the state space is $S = \{0, 1, 2\}$. The transition matrix is $\underline{\underline{P}} = \begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 1/2 & 1/3 & 1/6 \\ 0 & 1/2 & 1/2 \end{pmatrix}$

   (b) Let $f(x)$ be the expected number of clients that the barber has to serve until the number of clients in line reaches zero. Of course $f(0) = 0$ and we have

   $$f(2) = 1 + \frac{1}{2}f(1) + \frac{1}{2}f(2), \qquad f(1) = 1 + \frac{1}{2}0 + \frac{1}{3}f(1) + \frac{1}{6}f(2).$$

   Solving these equations we obtain $f(1) = 8/3$ and $f(2) = 14/3$. Thus the expected time (in minutes) until the barber can take a break is $20\frac{14}{3} = \frac{280}{3} = 93.33$