Midterm Exam - October 27, 2017, Stochastic Processes

1. On the morning of the n'th workday Y_n faulty cars arrive to the garage of an auto mechanic, where $Y_1, Y_2, \dots$ are i.i.d. with $\mathbb{P}(Y_i = 0) = \frac{2}{5}$, $\mathbb{P}(Y_i = 1) = \frac{1}{3}$, $\mathbb{P}(Y_i = 2) = \frac{4}{15}$. He can repair one car per day.

   (a) What is the average number of evenings per year when there are no cars waiting to be repaired in his garage?

   (b) Yesterday evening there were no cars waiting to be repaired in his garage. What is the expected number of workdays until the next evening without any car to be repaired?

   (c) If there are at least two cars in his garage in the evening then he becomes worried about work overload. Yesterday evening there were no cars waiting to be repaired in his garage. What is the expected number of worried evenings until the next evening without any car in the garage?

   Solution: Let X_n denote the number of cars in the garage on the n'th evening. Then (X_n) is a birth-death process with $q_x \equiv \frac{2}{5}$ for any $x \in \{1, 2, 3, \dots\}$ and $p_x \equiv \frac{4}{15}$ for any $x \in \{0, 1, 2, \dots\}$.

   (a) The stationary distribution is GEO$(1 - \frac{4/15}{2/5})$, that is $\pi_x = \frac{1}{3}\left(\frac{2}{3}\right)^x$, $x = 0, 1, 2, \dots$, see page 82. In particular, the average number of evenings per year when there are no cars in his garage is $365\pi_0 = 365/3$.
   Another way to see this: The expected number of cars that the auto mechanic repairs this year is roughly equal to $365 \cdot \mathbb{E}(Y_i) = 356 \cdot \frac{13}{15}$, which is equal to the average number of days when he repairs a car. Thus the expected number of days when he does not repair a car is $356 \cdot \frac{2}{15}$. This number is also equal to $365\pi_0\frac{2}{5}$ (he does not repair a car today if and only if there was no car in the garage yesterday evening and no car arrived today), thus $365\pi_0 = 356 \cdot \frac{2}{15}\frac{5}{2} = 365/3$.

   (b) The expected number of days until the next evening with an empty garage is $1/\pi_0 = 3$, see page 87.

   (c) We know from page 87 that $E_0\left(\sum_{n=0}^{T_0-1} \mathbb{1}[X_n = x]\right) = \pi_x/\pi_0 = \left(\frac{2}{3}\right)^x$, therefore the expected number of worried evenings until the next evening without any car in the garage is

$$E_0\left(\sum_{n=0}^{T_0-1} \mathbb{1}[X_n \geq 2]\right) = E_0\left(\sum_{n=0}^{T_0-1}\sum_{x=2}^{\infty} \mathbb{1}[X_n = x]\right) =$$
$$\sum_{x=2}^{\infty} E_0\left(\sum_{n=0}^{T_0-1} \mathbb{1}[X_n = x]\right) = \sum_{x=2}^{\infty}\left(\frac{2}{3}\right)^x = \frac{(2/3)^2}{1-2/3} = 4/3.$$

2. Find $\lim_{n\to\infty} \underline{\underline{P}}^n$, where

$$\underline{\underline{P}} = \begin{pmatrix} 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/3 & 1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

   Solution: The states 1 and 2 are transient, while 4 is an absorbing state and $\{3, 5\}$ is a closed irreducible component.

   Let $h(x) := P_x(T_{\{3,5\}} < T_4)$, where $x = 1$ or $x = 3$. We have $h(1) = \frac{1}{3}\cdot 1 + \frac{1}{3}h(1) + \frac{1}{3}h(2)$ and $h(2) = \frac{1}{3}h(1) + \frac{1}{3}h(2) + \frac{1}{3}\cdot 0$, and solving this system of linear equations we obtain $h(1) = 2/3$ and $h(2) = 1/3$.

   We also need to find the stationary distribution (π_3, π_5) on the closed irreducible component $\{3, 5\}$: $\pi_3 = \frac{1}{2}\pi_3 + \pi_5$, $\pi_5 = \frac{1}{2}\pi_3$ and $\pi_3 + \pi_5 = 1$. Solving this system of linear equations we obtain $\pi_3 = 2/3$ and $\pi_5 = 1/3$.

   Putting things together we obtain

$$\lim_{n\to\infty} \underline{\underline{P}}^n = \begin{pmatrix} 0 & 0 & h(1)\pi_3 & 1-h(1) & h(1)\pi_5 \\ 0 & 0 & h(2)\pi_3 & 1-h(2) & h(2)\pi_5 \\ 0 & 0 & \pi_3 & 0 & \pi_5 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \pi_3 & 0 & \pi_5 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 4/9 & 1/3 & 2/9 \\ 0 & 0 & 2/9 & 2/3 & 1/9 \\ 0 & 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2/3 & 0 & 1/3 \end{pmatrix}$$