Midterm Exam - December 1, 2017, Stochastic Processes

1. Cars arrive at a gas station according to a Poisson point process. The average waiting time between two consecutive arrivals is 2 minutes. One third of these cars are Jeeps, two third of them are SUV's. Jeeps are diesel-powered, while half of the SUV's are diesel-powered, half of them are gasoline-powered.

   (a) Given that 12 cars arrived between 10.15 and 10.35, what is the expectation and variance of the number of cars that arrive between 10.25 and 10.55?

   (b) Given that 6 cars bought diesel between 11.15 and 11.35, what is the probability that at least two of them were Jeeps?

   (c) I start observing the cars at 11.50. What is the expected number of SUV arrivals before the first Jeep arrival?

   Solution:

   (a) If 12 cars arrived between 10.15 and 10.35, then the number of cars that arrive between 10.25 and 10.35 is $\mathrm{BIN}(12, \frac{1}{2})$ distributed. One has to add to this the number of cars that arrive between 10.35 and 10.55, which is independent with $\mathrm{POI}(10)$ distribution. Thus the desired expectation is $12 \cdot \frac{1}{2} + 10 = 16$ and the desired variance is $12 \cdot \frac{1}{2} \cdot (1 - \frac{1}{2}) + 10 = 13$.

   (b) The cars arrive according to a $\mathrm{PPP}(\frac{1}{2})$. Coloring: Jeeps: $\mathrm{PPP}(\frac{1}{6})$, SUV's: $\mathrm{PPP}(\frac{1}{3})$, independent. Coloring: diesel Jeeps: $\mathrm{PPP}(\frac{1}{6})$, diesel SUV's: $\mathrm{PPP}(\frac{1}{6})$, gasoline SUV's: $\mathrm{PPP}(\frac{1}{6})$, independent. Thus a diesel-powered car is a Jeep with probability $\frac{1}{2}$. Given 6 diesel-powered cars, the number of Jeeps has $\mathrm{BIN}(6, \frac{1}{2})$ distribution. Thus the probability that at least two of those 6 were Jeeps is $1 - \left(\frac{1}{2}\right)^6 - 6\left(\frac{1}{2}\right)^6 = 1 - \frac{7}{64} = \frac{57}{64}$.

   (c) The distribution of SUV arrivals before the first Jeep arrival is pessimistic $\mathrm{GEO}(\frac{1}{3})$, thus the expectation is $\frac{1-\frac{1}{3}}{\frac{1}{3}} = 2$. This is intuitively clear: since there are twice as many SUV's as Jeeps, the average number of SUV's between two Jeeps is 2.

2. There are two tennis courts. Pairs of tennis players arrive according to a Poisson point process at rate 2 per hour and play for an exponentially distributed amount of time with mean 1/2 hours. If there is already one pair of players waiting for a court to become available then new arrivals will leave. If currently both courts are occupied but there are no players waiting, what is the expected time until both courts become empty?

   Solution: Denote by X_t the number of pairs of tennis players playing or waiting at time t. Then (X_t) is a birth and death process with state space $S = \{0, 1, 2, 3\}$, arrival rates $\lambda_0 = \lambda_1 = \lambda_2 = 2$ and departure rates $\mu_1 = 2$, $\mu_2 = \mu_3 = 4$. Denote by $f(x) = E_x(T_0)$, the expected value of $T_0 = \min\{\, t \,:\, X_t = 0 \,\}$ (the time when the tennis court first becomes empty), given that we start with $X_0 = x$. We want to find $f(2)$. We can write down the system of equations

$$f(3) = \frac{1}{4} + f(2), \quad f(2) = \frac{1}{6} + \frac{1}{3} f(3) + \frac{2}{3} f(1), \quad f(1) = \frac{1}{4} + \frac{1}{2} f(2).$$

   Substituting the first and third equation into the second one we obtain

$$f(2) = \frac{1}{6} + \frac{1}{3}\left(\frac{1}{4} + f(2)\right) + \frac{2}{3}\left(\frac{1}{4} + \frac{1}{2} f(2)\right).$$

   Solving this we find $f(2) = \frac{5}{4}$.