Stoch. Anal. HW assignment 10. Solutions

1. *Kolmogorov's backward equation.* Let (X_t) denote the solution of the SDE

$$\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t.$$

(Let us assume that both $\mu : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ are Lipschitz-continuous.)
Let us also consider the solution $u(t,x)$ of the parabolic PDE

$$\partial_t u(t,x) = \mu(x)\partial_x u(t,x) + \frac{1}{2}\sigma^2(x)\partial_{xx}u(t,x), \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \tag{1}$$

$$u(0,x) = f(x), \qquad x \in \mathbb{R}. \tag{2}$$

(Let us assume that all functions involved are bounded and continuous.)
Show that for each $T \geq 0$, the following probabilistic interpretation of u holds:

$$u(T,x) = \mathbb{E}(f(X_T) \,|\, X_0 = x).$$

Hint: For any given T, consider

$$M_t := u(T-t, X_t), \qquad 0 \leq t \leq T.$$

Solution: We will first show that $(M_t), 0 \leq t \leq T$ is a martingale.
Let us calculate the stochastic differential of (M_t) using the time-dependent Itô formula:

$$\begin{aligned}
\mathrm{d}M_t = \mathrm{d}u(T-t,X_t) = -u_t(T-t,X_t)\mathrm{d}t + u_x(T-t,X_t)\mathrm{d}X_t + \frac{1}{2}u_{xx}(T-t,X_t)\mathrm{d}[X]_t = \\
- u_t(T-t,X_t)\mathrm{d}t + u_x(T-t,X_t)(\mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t) + \frac{1}{2}u_{xx}(T-t,X_t)\sigma^2(X_t)\mathrm{d}t = \\
\left(-u_t(T-t,X_t) + \mu(X_t)u_x(T-t,X_t) + \frac{1}{2}\sigma^2(X_t)u_{xx}(T-t,X_t)\right)\mathrm{d}t + \sigma(X_t)u_x(T-t,X_t)\mathrm{d}B_t \overset{(1)}{=} \\
0\,\mathrm{d}t + \sigma(X_t)u_x(T-t,X_t)\mathrm{d}B_t.
\end{aligned}$$

Thus the drift term of $\mathrm{d}M_t$ vanishes, and since $(M_t)_{t=0}^T$ is bounded, it is a martingale.
Let us denote by $\mathbb{E}_x(\,\cdot\,) = \mathbb{E}(\,\cdot \,|\, X_0 = x)$.

$$u(T,x) = \mathbb{E}_x(u(T-0,X_0)) = \mathbb{E}_x(M_0) = \mathbb{E}_x(M_T) = \mathbb{E}_x(u(T-T,X_T)) \overset{(2)}{=} \mathbb{E}_x(f(X_T)).$$

2. Use the result of the previous exercise to show that the solution of the PDE

$$\partial_t u(t,x) = \alpha x \partial_x u(t,x) + \frac{1}{2}\beta^2 x^2 \partial_{xx} u(t,x), \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}_+, \tag{3}$$

$$u(0,x) = f(x), \qquad x \in \mathbb{R} \tag{4}$$

is equal to

$$u(t,x) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} f\big(x \exp\left\{\beta y + (\alpha - \beta^2/2)t\right\}\big) e^{-\frac{y^2}{2t}} \mathrm{d}y, \qquad t, x > 0.$$

Solution: Here $\mu(x) = \alpha \cdot x$ and $\sigma(x) = \beta \cdot x$, thus the SDE is $\mathrm{d}X_t = \alpha X_t \mathrm{d}t + \beta X_t \mathrm{d}B_t$. This is the SDE of geometric Brownian motion, thus the strong solution of the SDE is

$$X_t = X_0 \cdot \exp\left(\beta B_t + (\alpha - \frac{1}{2}\beta^2)t\right), \qquad t \geq 0, \tag{5}$$

see Lecture 24. Thus we have

$$u(t,x) \overset{(*)}{=} \mathbb{E}_x[f(X_t)] \overset{(5)}{=} \mathbb{E}\left[f\left(x \cdot \exp\left(\beta B_t + (\alpha - \frac{1}{2}\beta^2)t\right)\right)\right] \overset{(\bullet)}{=} \int_{-\infty}^{\infty} f\Big(x \exp\left\{\beta y + (\alpha - \beta^2/2)t\right\}\Big) \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} \mathrm{d}y,$$

where in $(*)$ we used the result of the previous exercise and in $(\bullet)$ we used the law of unconscious statistician together with the fact that $B_t \sim \mathcal{N}(0,t)$.

3. *Kolmogorov's forward equation.* Let (X_t) denote the solution of the SDE $\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t$, where μ and σ are Lipschitz. The solution is a time-homogeneous Markov process, so let us consider its transition probability density function $p_t(x,y)$, $t \geq 0, x, y \in \mathbb{R}$ (see lecture 10). Show that for any $x \in \mathbb{R}$, the following PDE is satisfied:

$$\frac{\partial}{\partial t} p_t(x,y) = -\frac{\partial}{\partial y}(\mu(y) p_t(x,y)) + \frac{1}{2}\frac{\partial^2}{\partial y^2}(\sigma^2(y) p_t(x,y)), \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}. \tag{6}$$

Instruction: You may assume that all functions involved are sufficiently regular for your calculations to make sense without giving technical details.

Solution: Following the ideas of lecture 32, let us take an arbitrary compactly supported smooth test function $f : \mathbb{R} \to \mathbb{R}$. Let us denote by $\mathbb{E}_x(\,\cdot\,) = \mathbb{E}(\,\cdot \,|\, X_0 = x)$ and $\mathbb{P}_x(\,\cdot\,) = \mathbb{P}(\,\cdot \,|\, X_0 = x)$.

We have $p_t(x,y)\,\mathrm{d}y = \mathbb{P}_x(\, X_t \in [y, y+\mathrm{d}y]\,)$ (c.f. lecture 10).

By the law of unconscious statistician we have $\mathbb{E}_x[f(X_t)] = \int_{-\infty}^{\infty} f(y) p_t(x,y)\,\mathrm{d}y$.

Differentiating this identity with respect to t, we obtain

$$\frac{\partial}{\partial t}\mathbb{E}_x[f(X_t)] = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} f(y) p_t(x,y)\,\mathrm{d}y = \int_{-\infty}^{\infty} f(y)\frac{\partial}{\partial t} p_t(x,y)\,\mathrm{d}y. \tag{7}$$

We can calculate the same quantity differently:

$$\frac{\partial}{\partial t}\mathbb{E}_x[f(X_t)] = \lim_{\mathrm{d}t\to 0}\frac{\mathbb{E}_x[f(X_{t+\mathrm{d}t})] - \mathbb{E}_x[f(X_t)]}{\mathrm{d}t} = \lim_{\mathrm{d}t\to 0}\frac{\mathbb{E}_x[f(X_{t+\mathrm{d}t}) - f(X_t)]}{\mathrm{d}t} = \lim_{\mathrm{d}t\to 0}\frac{\mathbb{E}_x[\mathrm{d}f(X_t)]}{\mathrm{d}t}. \tag{8}$$

By Itô's formula we have

$$\mathrm{d}f(X_t) = f'(X_t)\mathrm{d}X_t + \frac{1}{2}f''(X_t)\mathrm{d}[X]_t = \left(f'(X_t)\mu(X_t) + \frac{1}{2}f''(X_t)\sigma^2(X_t)\right)\mathrm{d}t + f'(X_t)\sigma(X_t)\mathrm{d}B_t. \tag{9}$$

Noting that $\mathbb{E}_x[\mathrm{d}B_t \,|\, \mathcal{F}_t] = \mathbb{E}[\mathrm{d}B_t] = 0$, we obtain

$$\mathbb{E}_x[f'(X_t)\sigma(X_t)\mathrm{d}B_t] = \mathbb{E}_x[\,\mathbb{E}_x[f'(X_t)\sigma(X_t)\mathrm{d}B_t \,|\, \mathcal{F}_t]\,] = \mathbb{E}_x[\,f'(X_t)\sigma(X_t)\mathbb{E}_x[\mathrm{d}B_t \,|\, \mathcal{F}_t]\,] = 0, \tag{10}$$

therefore

$$\mathbb{E}_x[\mathrm{d}f(X_t)] \overset{(9),(10)}{=} \mathbb{E}_x\left[f'(X_t)\mu(X_t) + \frac{1}{2}f''(X_t)\sigma^2(X_t)\right]\mathrm{d}t. \tag{11}$$

From these observations (somewhat non-rigorously) we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\mathbb{E}_x[f(X_t)] \overset{(8),(11)}{=} \mathbb{E}_x\left[f'(X_t)\mu(X_t) + \frac{1}{2}f''(X_t)\sigma^2(X_t)\right] = \\
&\int_{-\infty}^{\infty}\left(f'(y)\mu(y) + \frac{1}{2}f''(y)\sigma^2(y)\right)p_t(x,y)\,\mathrm{d}y = \int_{-\infty}^{\infty} f'(y)\mu(y)p_t(x,y)\,\mathrm{d}y + \int_{-\infty}^{\infty}\frac{1}{2}f''(y)\sigma^2(y)p_t(x,y)\,\mathrm{d}y.
\end{aligned} \tag{12}$$

Integrating by parts and using that f is compactly supported, we obtain

$$\int_{-\infty}^{\infty} f'(y)\mu(y)p_t(x,y)\,\mathrm{d}y = -\int_{-\infty}^{\infty} f(y)\frac{\partial}{\partial y}\left(\mu(y)p_t(x,y)\right)\mathrm{d}y. \tag{13}$$

Integrating by parts twice and using that f is compactly supported, we obtain

$$\int_{-\infty}^{\infty}\frac{1}{2}f''(y)\sigma^2(y)p_t(x,y)\,\mathrm{d}y = \int_{-\infty}^{\infty} f(y)\frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2(y)p_t(x,y)\right)\mathrm{d}y. \tag{14}$$

Putting our two different expressions (i.e., (7) and (12)) for $\frac{\partial}{\partial t}\mathbb{E}_x[f(X_t)]$ together we obtain

$$\int_{-\infty}^{\infty} f(y)\frac{\partial}{\partial t}p_t(x,y)\,\mathrm{d}y = \int_{-\infty}^{\infty} f(y)\left(-\frac{\partial}{\partial y}(\mu(y)p_t(x,y)) + \frac{1}{2}\frac{\partial^2}{\partial y^2}(\sigma^2(y)p_t(x,y))\right)\mathrm{d}y.$$

Since this identity is true for any test function f, we obtain the desired (6).

4. *Poisson's equation with Dirichlet boundary condition.* Given a compact domain $\mathcal{D} \subseteq \mathbb{R}^d$ with a smooth boundary $\partial\mathcal{D}$, moreover $g : \mathcal{D} \to \mathbb{R}$ and $f : \partial\mathcal{D} \to \mathbb{R}$ (both bounded and continuous), we say that the function $u : \mathcal{D} \to \mathbb{R}$ is the solution of Poisson's equation with Dirichlet boundary condition if $\frac{1}{2}\Delta u(\underline{x}) = g(\underline{x})$ for all $\underline{x} \in \mathcal{D}^o$ (the interior of $\mathcal{D}$) and $u(\underline{x}) = f(\underline{x})$ for all $\underline{x} \in \partial\mathcal{D}$. Apply the optional stopping theorem to an appropriately chosen martingale to show that the solution u has the following probabilistic representation:

$$u(\underline{x}) = \mathbb{E}\left(f(\underline{B}_\tau) - \int_0^\tau g(\underline{B}_s)\,\mathrm{d}s \,\middle|\, \underline{B}_0 = \underline{x} \right), \qquad \underline{x} \in \mathcal{D}, \tag{15}$$

where $(\underline{B}_t)$ is a d-dim. Brownian motion and $\tau = \min\{\, t : \underline{B}_t \in \partial\mathcal{D}\}$ is the exit time of $(\underline{B}_t)$ from $\mathcal{D}^o$.

Solution: Let $\mathbb{E}_{\underline{x}} = \mathbb{E}(\,\cdot \,|\, \underline{B}_0 = \underline{x}\,)$.

Let

$$M_t = u(\underline{B}_t) - \int_0^t g(\underline{B}_s)\,\mathrm{d}s.$$

First we show that (M_t) is a martingale using the multivariable Itô formula (see lecture 23):

$$\mathrm{d}M_t = \mathrm{d}u(\underline{B}_t) - g(\underline{B}_t)\,\mathrm{d}t = \sum_{i=1}^d u_{x_i}(\underline{B}_t)\mathrm{d}B_i(t) + \frac{1}{2}\Delta u(\underline{B}_t)\mathrm{d}t - g(\underline{B}_t)\mathrm{d}t = \sum_{i=1}^d u_{x_i}(\underline{B}_t)\mathrm{d}B_i(t), \tag{16}$$

where the coefficients of $\mathrm{d}t$ cancelled each other out since $\frac{1}{2}\Delta u(\underline{x}) - g(\underline{x}) \equiv 0$ by the PDE.

The drift term of $\mathrm{d}M_t$ vanishes and one can show that (M_t) is indeed a martingale (we omit the details).

One can check that the optional stopping theorem can be applied to our martingale (M_t) and stopping time τ (defined as above) using the same method as in the case of Dynkin's formula (see lecture 35).

Thus we have $\mathbb{E}_{\underline{x}}(M_\tau) = \mathbb{E}_{\underline{x}}(M_0)$ by the OST, and we have $\mathbb{E}_{\underline{x}}(M_0) = u(\underline{x})$, while

$$\mathbb{E}_{\underline{x}}(M_\tau) = \mathbb{E}_{\underline{x}}\left(u(\underline{B}_\tau) - \int_0^\tau g(\underline{B}_s)\,\mathrm{d}s \right) = \mathbb{E}_{\underline{x}}\left(f(\underline{B}_\tau) - \int_0^\tau g(\underline{B}_s)\,\mathrm{d}s \right),$$

where the last equation holds since $u(\underline{x}) = f(\underline{x})$ for all $\underline{x} \in \partial\mathcal{D}$ and $\underline{B}_\tau \in \partial\mathcal{D}$.

Putting these together we obtain the desired (15).