Stoch. Anal. HW assignment 1. 2025, Solutions

1. Let (X, Y) denote a pair of continuous random variables with joint density function $f(x, y)$.

   Assume that $\mathbb{E}(|Y|) < +\infty$. Let $f_X(x) = \int_{-\infty}^{\infty} f(x, y)\,\mathrm{d}y$ and define

   $$H(x) = \int_{-\infty}^{\infty} \frac{f(x, y)}{f_X(x)} y\,\mathrm{d}y. \tag{1}$$

   Prove that the random variable $H(X)$ satisfies the abstract definition of $\mathbb{E}(\,Y \,|\, \sigma(X)\,)$.

   Hint: You have to check that $Z := H(X)$ satisfies the following properties: $\mathbb{E}(|H(X)|) < +\infty$, $H(X)$ is $\sigma(X)$-measurable and that $\mathbb{E}(Y\mathbb{1}_A) = \mathbb{E}(H(X)\mathbb{1}_A)$ for any $A \in \sigma(X)$. You can use that $A \in \sigma(X)$ if and only if $A = X^{-1}(B)$ for some Borel set $B \in \mathcal{B}(\mathbb{R})$. Use the „law of the unconscious statistician" which states that if the probability density function (p.d.f.) of Z is $g(z)$, then $\mathbb{E}(\varphi(Z)) = \int_{-\infty}^{\infty} \varphi(z)g(z)\,\mathrm{d}z$ for any function $\varphi : \mathbb{R} \to \mathbb{R}$ for which the integral makes sense.

   Solution: If we want to check that the random variable $H(X)$ satisfies the abstract definition of $\mathbb{E}(\,Y \,|\, \sigma(X)\,)$, we need to check three properties.

   First: indeed $H(X)$ is $\sigma(X)$-measurable, since it is a deterministic function of the random variable X.

   Second: now we show that $\mathbb{E}(|H(X)|) < +\infty$:

   $$\mathbb{E}(|H(X)|) = \int_{-\infty}^{\infty} |H(x)| f_X(x)\mathrm{d}x \overset{(1)}{=} \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \frac{f(x, y)}{f_X(x)} y\,\mathrm{d}y \right| f_X(x)\mathrm{d}x \leq$$
   $$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(x, y)}{f_X(x)} |y|\,\mathrm{d}y f_X(x)\mathrm{d}x = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)|y|\,\mathrm{d}y\,\mathrm{d}x = \mathbb{E}(|Y|) < +\infty.$$

   Third: we now show that $\mathbb{E}(H(X)\mathbb{1}_A) = \mathbb{E}(Y\mathbb{1}_A)$ for any $\sigma(X)$-measurable event A. Any such event is of form $A = \{X \in B\}$, where B is a Borel subset of $\mathbb{R}$. So we calculate

   $$\mathbb{E}(Y\mathbb{1}_A) = \mathbb{E}(Y\mathbb{1}[X \in B]) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) y \mathbb{1}[x \in B]\,\mathrm{d}y\,\mathrm{d}x = \int_B \int_{-\infty}^{\infty} f(x, y) y\,\mathrm{d}y\,\mathrm{d}x$$

   and

   $$\mathbb{E}(H(X)\mathbb{1}_A) = \mathbb{E}(H(X)\mathbb{1}[X \in B]) = \int_{-\infty}^{\infty} H(x)\mathbb{1}[x \in B] f_X(x)\,\mathrm{d}x = \int_B H(x) f_X(x)\,\mathrm{d}x \overset{(1)}{=}$$
   $$\int_B \int_{-\infty}^{\infty} \frac{f(x, y)}{f_X(x)} y\,\mathrm{d}y\, f_X(x)\,\mathrm{d}x = \int_B \int_{-\infty}^{\infty} f(x, y) y\,\mathrm{d}y\,\mathrm{d}x.$$

   Thus indeed $\mathbb{E}(H(X)\mathbb{1}_A) = \mathbb{E}(Y\mathbb{1}_A)$ for any $\sigma(X)$-measurable event A. Thus $H(X)$ satisfies the abstract definition of $\mathbb{E}(\,Y \,|\, \sigma(X)\,)$, so by the uniqueness of random variables satisfying this definition, we obtain $H(X) = \mathbb{E}(\,Y \,|\, \sigma(X)\,)$, as desired.

2. Let X and Y denote the outcomes of two i.i.d. die rolls. Let $Z := X + Y$. Find $\mathbb{E}(Z \mid X)$ and $\mathbb{E}(X \mid Z)$.

Solution:

$$\mathbb{E}(Z \mid X) = \mathbb{E}(X + Y \mid X) \overset{\text{linearity}}{=} \mathbb{E}(X \mid X) + \mathbb{E}(Y \mid X) \overset{(*)}{=} X + \mathbb{E}(Y) = X + \frac{7}{2},$$

where in $(*)$ we used that X is measurable w.r.t. the sigma-algebra generated by X and also used that Y and X are independent.

As for $\mathbb{E}(X \mid Z)$, let us note that $\mathbb{P}(X = k \,|Z = \ell) = \mathbb{P}(Y = k \,|Z = \ell)$ for any k and l, since the two die rolls are i.i.d. and Z is a symmetric function of X and Y. Thus $\mathbb{E}(X \mid Z) = \mathbb{E}(Y \mid Z)$ and

$$Z = \mathbb{E}(Z \mid Z) = \mathbb{E}(X + Y \mid Z) \overset{\text{linearity}}{=} \mathbb{E}(X \mid Z) + \mathbb{E}(Y \mid Z) = 2\mathbb{E}(X \mid Z),$$

thus $\mathbb{E}(X \mid Z) = Z/2$.

3. The moment generating function $M : \mathbb{R} \to (0, \infty]$ of a random variable X is defined by

$$M(\lambda) = M_X(\lambda) = \mathbb{E}(e^{\lambda X}).$$

If we assume that there exists $\lambda_0 > 0$ such that $M(\lambda_0) < \infty$ and $M(-\lambda_0) < \infty$ then it is known that

$$\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} M(\lambda) = \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \mathbb{E}(e^{\lambda X}) \overset{(*)}{=} \mathbb{E}(\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} e^{\lambda X}) = \mathbb{E}(X^n e^{\lambda X}) \quad \text{for any} \quad \lambda \in (-\lambda_0, \lambda_0).$$

(The only non-trivial equation is marked by $(*)$: if you want to see a proof of the fact that expectation and differentiation can be interchanged in this case, write an e-mail to me.)

Note that this implies $\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} M(\lambda)\big|_{\lambda=0} = \mathbb{E}(X^n)$, hence the name „moment generating function".

(a) Let $X \sim \mathcal{N}(0, 1)$ (standard normal). Calculate $M_X(\lambda)$.
Hint: Use the „law of the unconscious statistician". You should also use that any integral of form $\int_{-\infty}^{\infty} \exp(-ax^2 + bx + c)\,\mathrm{d}x$ can be calculated by manipulating the identity $\int_{-\infty}^{\infty} g(x)\,\mathrm{d}x = 1$, where $g(x)$ is the p.d.f. of some random variable $\mathcal{N}(\mu, \sigma^2)$.

(b) Let $Y \sim \mathcal{N}(\mu, \sigma^2)$ (normal random variable with expectation μ and variance σ^2). Calculate $M_Y(\lambda)$.
Hint: You can do this without calculating any further integrals: you just have to use part (a) of this exercise and the fact that Y has the same distribution as $\sigma X + \mu$.

(c) Let $Z \sim \mathcal{N}(0, \sigma^2)$. Use the Taylor expansion of $M_Z(\lambda)$ about $\lambda = 0$ to calculate $\mathbb{E}(Z^n)$, $n \geq 0$.
Hint: You will only have to use the Taylor expansion of the exponential function in a clever way.

(d) Let $Z \sim \mathcal{N}(0, \sigma^2)$. Let $X = Z^2$. Calculate the variance of X using the results of part (c).

Solution:

(a) Let $X \sim \mathcal{N}(0, 1)$ and calculate the moment generating function

$$M_X(\lambda) = \mathbb{E}(e^{\lambda X}) = \int_{-\infty}^{\infty} e^{\lambda x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,\mathrm{d}x = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\lambda)^2 + \frac{1}{2}\lambda^2}\,\mathrm{d}x =$$
$$e^{\frac{1}{2}\lambda^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\lambda)^2}\,\mathrm{d}x = e^{\frac{1}{2}\lambda^2},$$

because $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\lambda)^2}$ is the p.d.f. of $\mathcal{N}(\lambda, 1)$, thus $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$.

(b) If $Y = \sigma X + \mu$, then $Y \sim \mathcal{N}(\mu, \sigma^2)$, so the moment generating function of Y can be calculated using the moment generating function of X:

$$M_Y(\lambda) = \mathbb{E}(e^{\lambda Y}) = \mathbb{E}(e^{\lambda(\sigma X + \mu)}) = e^{\lambda\mu} \mathbb{E}(e^{(\lambda\sigma)X}) = e^{\lambda\mu} e^{\frac{1}{2}(\lambda\sigma)^2} = e^{\lambda\mu + \frac{1}{2}\lambda^2\sigma^2}.$$

(c) $Z \sim \mathcal{N}(0, \sigma^2)$, so $M_Z(\lambda) = e^{\frac{1}{2}\lambda^2\sigma^2}$. Now $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, so substituting $x = \frac{1}{2}\lambda^2\sigma^2$ gives $M_Z(\lambda) = \sum_{k=0}^{\infty} \frac{\sigma^{2k}\lambda^{2k}}{2^k k!}$, this is the power series expansion of $M_Z(\lambda)$ about $\lambda = 0$.
If n is odd, then the coefficient of λ^n is zero in this expansion, so $M_Z^{(n)}(0) = 0$, so $\mathbb{E}(Z^n) = 0$.
If n is even, then $n = 2k$, and the coefficient of λ^n is $\frac{\sigma^n}{2^{n/2}(n/2)!}$. On the other hand, the coefficient of λ^n in the Taylor series is $\frac{M_Z^{(n)}(0)}{n!}$, thus $\mathbb{E}(Z^n) = M_Z^{(n)}(0) = \frac{\sigma^n n!}{2^{n/2}(n/2)!}$. This can also be expressed using a so-called „semifactorial", see internet.

(d) $\mathrm{Var}(X) = \mathrm{Var}(Z^2) = \mathbb{E}(Z^4) - \mathbb{E}(Z^2)^2 = \frac{\sigma^4 4!}{2^{2 \cdot 2!}} - (\sigma^2)^2 = 3\sigma^4 - \sigma^4 = 2\sigma^4$

4. Let (X, Y) denote a pair of continuous random variables with joint density function

$$f(x,y) = c \exp\left(-\frac{5}{2}x^2 - 3xy - y^2\right).$$

(a) Find c.
Hint: We must have $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,\mathrm{d}x\mathrm{d}y = 1$. You should also use the fact that any integral of form $\int_{-\infty}^{\infty} \exp(-ax^2 + bx + c)\,\mathrm{d}x$ can be calculated by manipulating the identity $\int_{-\infty}^{\infty} g(x)\,\mathrm{d}x = 1$, where $g(x)$ is the p.d.f. of some random variable $\mathcal{N}(\mu, \sigma^2)$.

(b) Find the density function $f_X(x)$ of X. *Hint:* $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,\mathrm{d}y$.

(c) Find the covariance $\mathrm{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$. *Hint:* Bivariate version of the „law of the unconscious statistician": $\mathbb{E}(\varphi(X,Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(x,y) f(x,y)\,\mathrm{d}x\mathrm{d}y$ for any $\varphi : \mathbb{R}^2 \to \mathbb{R}$.

(d) Find $\mathbb{E}(\,Y \,|\, \sigma(X)\,)$. *Hint:* You can use that $\mathbb{E}(\,Y \,|\, \sigma(X)\,) = H(X)$, where $H(x)$ is given above.

Solution: One can solve this exercise without calculating integrals (but relying on the theory of *multivariate normal distribution*). We will discuss that solution in class. Here we give the „pedestrian" solution (which involves integrals but makes no reference to multivariate normal distribution).

(a)

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,\mathrm{d}y = ce^{-\frac{5}{2}x^2}\int_{-\infty}^{\infty} \exp\left(-3xy - y^2\right)\,\mathrm{d}y =$$
$$ce^{-\frac{5}{2}x^2}\int_{-\infty}^{\infty} \exp\left(-(y+\frac{3}{2}x)^2 + \frac{9}{4}x^2\right)\,\mathrm{d}y = ce^{-\frac{1}{4}x^2}\int_{-\infty}^{\infty} \exp\left(-\frac{(y+\frac{3}{2}x)^2}{2(1/\sqrt{2})^2}\right)\,\mathrm{d}y =$$
$$ce^{-\frac{1}{4}x^2}(1/\sqrt{2})\sqrt{2\pi}\int_{-\infty}^{\infty} \frac{1}{(1/\sqrt{2})\sqrt{2\pi}} \exp\left(-\frac{(y+\frac{3}{2}x)^2}{2(1/\sqrt{2})^2}\right)\,\mathrm{d}y \overset{(*)}{=}$$
$$ce^{-\frac{1}{4}x^2}(1/\sqrt{2})\sqrt{2\pi} = ce^{-\frac{x^2}{2(\sqrt{2})^2}}(1/\sqrt{2})\sqrt{2\pi},$$

where in $(*)$ we used $\int_{-\infty}^{\infty} f(y)\,\mathrm{d}y = 1$, where $f(y)$ is the p.d.f. of $\mathcal{N}(-\frac{3}{2}x, (\frac{1}{\sqrt{2}})^2)$. Now we see that $X \sim \mathcal{N}(0, (\sqrt{2})^2)$, therefore $f_X(x) = \frac{1}{\sqrt{2}\sqrt{2\pi}} e^{-\frac{x^2}{2(\sqrt{2})^2}}$, thus $c \cdot (1/\sqrt{2})\sqrt{2\pi} = \frac{1}{\sqrt{2}\sqrt{2\pi}}$, thus $c = \frac{1}{2\pi}$.

(b) We have already found it.

(c) We already know that $X \sim \mathcal{N}(0, (\sqrt{2})^2)$, so $\mathbb{E}(X) = 0$, thus

$$\mathrm{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = \mathbb{E}(XY) - 0 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)\,\mathrm{d}y\,\mathrm{d}x =$$
$$\int_{-\infty}^{\infty} x\frac{1}{\sqrt{2}\sqrt{2\pi}} e^{-\frac{x^2}{2(\sqrt{2})^2}} \int_{-\infty}^{\infty} y \frac{1}{(1/\sqrt{2})\sqrt{2\pi}} \exp\left(-\frac{(y+\frac{3}{2}x)^2}{2(1/\sqrt{2})^2}\right)\,\mathrm{d}y\,\mathrm{d}x \overset{(**)}{=}$$
$$\int_{-\infty}^{\infty} x\frac{1}{\sqrt{2}\sqrt{2\pi}} e^{-\frac{x^2}{2(\sqrt{2})^2}} \left(-\frac{3}{2}x\right)\,\mathrm{d}x = -\frac{3}{2}\int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2}\sqrt{2\pi}} e^{-\frac{x^2}{2(\sqrt{2})^2}}\,\mathrm{d}x \overset{(***)}{=} -\frac{3}{2}\cdot(\sqrt{2})^2 = -3,$$

where in $(**)$ we used that $\int_{-\infty}^{\infty} yf(y)\,\mathrm{d}y = -\frac{3}{2}x$, where $f(y)$ is the p.d.f. of $\mathcal{N}(-\frac{3}{2}x, (\frac{1}{\sqrt{2}})^2)$ and in $(***)$ we used that $\int_{-\infty}^{\infty} x^2 g(x)\,\mathrm{d}x = (\sqrt{2})^2$, where $g(x)$ is the p.d.f. of $\mathcal{N}(0, (\sqrt{2})^2)$.

(d) The conditional density function of Y given $X = x$ is

$$f_{Y|X}(y\,|\,x) = \frac{f(x,y)}{f_X(x)} = \frac{\frac{1}{2\pi}\exp\left(-\frac{5}{2}x^2 - 3xy - y^2\right)}{\frac{1}{\sqrt{2}\sqrt{2\pi}}e^{-\frac{x^2}{4}}} = \frac{1}{(1/\sqrt{2})\sqrt{2\pi}} \exp\left(-\frac{(y+\frac{3}{2}x)^2}{2(1/\sqrt{2})^2}\right),$$

thus the conditional distribution of Y given $X = x$ is $\mathcal{N}(-\frac{3}{2}x, (\frac{1}{\sqrt{2}})^2)$, thus

$$H(x) = \int_{-\infty}^{\infty} y \cdot f_{Y|X}(y\,|\,x)\,\mathrm{d}y = -\frac{3}{2}x,$$

thus

$$\mathbb{E}(\,Y \,|\, \sigma(X)\,) = H(X) = -\frac{3}{2}X.$$