Stoch. Anal. HW assignment 2. Solutions

Note: Each of the 4 questions is worth 10 marks.

1. Let $S_n = \xi_1 + \cdots + \xi_n$, where $\xi_1, \xi_2, \ldots$, are i.i.d. and $\mathbb{P}(\xi_k = 1) = \mathbb{P}(\xi_k = -1) = \frac{1}{2}$, $k \geq 1$. That is: (S_n) is a one-dimensional simple, symmetric random walk. Let $(\mathcal{F}_n)_{n\geq 0}$ denote the natural filtration of (S_n).

   (a) Find the discrete Doob-Meyer decomposition of the process $(e^{\lambda S_n})_{n\geq 1}$ where $\lambda \in \mathbb{R}$, i.e., write $e^{\lambda S_n} = A_n + M_n$, where (A_n) is a predictable process and (M_n) is a martingale with zero expectation. Explicitly state the formula for A_n.
   Hint: You may want to use the notation $\cosh(\lambda) = \frac{1}{2}\left(e^{\lambda} + e^{-\lambda}\right)$.

   (b) Write M_n as the discrete stochastic integral $(H \cdot S)_n$ of a predictable process (H_n) with respect to the martingale (S_n). Explicitly state the formula for H_n.
   Hint: Observe that $S_k - S_{k-1}$ can only take two possible values. Notation: $\sinh(\lambda) = \frac{1}{2}\left(e^{\lambda} - e^{-\lambda}\right)$.

Solution:

(a) The definition of the natural filtration is:

$$\mathcal{F}_n = \sigma(S_1, \ldots, S_n) = \sigma(\eta_1, \ldots, \eta_n)$$

We have seen in class that A_n and M_n can be calculated with the following formulas:
$A_n = A_1 + \sum_{i=2}^{n} \Delta A_{i-1}$, where

$$\Delta A_{i-1} = \mathbb{E}(\Delta e^{\lambda S_{i-1}} \,|\, \mathcal{F}_{i-1}) = \mathbb{E}(e^{\lambda S_i} - e^{\lambda S_{i-1}} \,|\, \mathcal{F}_{i-1}) = \mathbb{E}(e^{\lambda S_i} \,|\, \mathcal{F}_{i-1}) - e^{\lambda S_{i-1}},$$

and $M_n = M_1 + \sum_{i=2}^{n} \Delta M_{i-1}$, where

$$\Delta M_{i-1} = \Delta e^{\lambda S_{i-1}} - \Delta A_{i-1} = \left(e^{\lambda S_i} - e^{\lambda S_{i-1}}\right) - \left(\mathbb{E}(e^{\lambda S_i} \,|\, \mathcal{F}_{i-1}) - e^{\lambda S_{i-1}}\right) = e^{\lambda S_i} - \mathbb{E}(e^{\lambda S_i} \,|\, \mathcal{F}_{i-1})$$

Moreover: $A_1 = \mathbb{E}(e^{\lambda S_1})$ and $M_1 = e^{\lambda S_1} - \mathbb{E}(e^{\lambda S_1})$. Putting things together, we obtain:

$$A_n = \mathbb{E}(e^{\lambda S_1}) + \sum_{i=2}^{n} (\mathbb{E}(e^{\lambda S_i} | \mathcal{F}_{i-1}) - e^{\lambda S_{i-1}}) = (\frac{1}{2}e^{\lambda} + \frac{1}{2}e^{-\lambda}) + \sum_{i=2}^{n} e^{\lambda S_{i-1}} (\frac{1}{2}e^{\lambda} + \frac{1}{2}e^{-\lambda} - 1),$$

$$M_n = \sum_{i=1}^{n} (e^{\lambda S_i} - \mathbb{E}(e^{\lambda S_i} | \mathcal{F}_{i-1})) = \sum_{i=1}^{n} (e^{\lambda S_i} - e^{\lambda S_{i-1}} (\frac{1}{2}e^{\lambda} + \frac{1}{2}e^{-\lambda})).$$

(Here we could write $\cosh(\lambda) = \frac{1}{2}e^{\lambda} + \frac{1}{2}e^{-\lambda}$ everywhere.)

(b) We want: $M_n = \sum_{i=1}^{n} H_i \cdot (S_i - S_{i-1}) = \sum_{i=1}^{n} H_i \xi_i$, where H_i is $\mathcal{F}_{i-1}$-measurable.
Taking the discrete differential of both sides, we want: $\Delta M_{n-1} = H_n \xi_n$ for any n.
Thus we want: $e^{\lambda S_n} - e^{\lambda S_{n-1}} (\frac{1}{2}e^{\lambda} + \frac{1}{2}e^{-\lambda}) = H_n \xi_n$. Now

$$e^{\lambda S_n} - e^{\lambda S_{n-1}} (\frac{1}{2}e^{\lambda} + \frac{1}{2}e^{-\lambda}) = e^{\lambda S_{n-1}} (e^{\lambda \xi_n} - \frac{1}{2}e^{\lambda} - \frac{1}{2}e^{-\lambda}) \overset{(*)}{=} e^{\lambda S_{n-1}} \frac{1}{2}(e^{\lambda} - e^{-\lambda}) \xi_n = e^{\lambda S_{n-1}} \sinh(\lambda) \xi_n,$$

where $(*)$ is an equality that holds in both of the cases $\xi_i = 1$ and $\xi_i = -1$. Indeed:

$$e^{\lambda \cdot (+1)} - \frac{1}{2}e^{\lambda} - \frac{1}{2}e^{-\lambda} = \frac{1}{2}(e^{\lambda} - e^{-\lambda}) \cdot (+1),$$
$$e^{\lambda \cdot (-1)} - \frac{1}{2}e^{\lambda} - \frac{1}{2}e^{-\lambda} = \frac{1}{2}(e^{\lambda} - e^{-\lambda}) \cdot (-1).$$

Thus $H_n = e^{\lambda S_{n-1}} \sinh(\lambda)$, which is indeed $\mathcal{F}_{n-1}$-measurable.

2. Let $\mathbb{F} = (\mathcal{F}_n)_{n=0}^{\infty}$ be a filtration and assume that $\mathcal{F}_0$ is the trivial sigma-field. Let $(N_n)_{n=1}^{\infty}$ be a square integrable martingale, i.e., assume that $\mathbb{E}(N_n^2) < \infty$ for each n.

   (a) Show that $\mathbb{E}(\Delta N_{i-1}^2 \,|\, \mathcal{F}_{i-1}) = \mathbb{E}(N_i^2 \,|\, \mathcal{F}_{i-1}) - N_{i-1}^2 = \mathbb{E}((\Delta N_{i-1})^2 \,|\, \mathcal{F}_{i-1})$.

   (b) Find the discrete Doob-Meyer decomposition of the process (N_n^2), i.e., write $N_n^2 = A_n + M_n$, where (A_n) is a predictable process and (M_n) is a martingale with zero expectation. Write down explicitly the formula for A_n.

   (c) Prove the *Pythagorean theorem for square-integrable martingales:*

$$\mathbb{E}(N_n^2) = \mathbb{E}(N_1^2) + \sum_{j=1}^{n-1} \mathbb{E}((\Delta N_j)^2)$$

   Hint: It is enough to take the expectation of both sides of the equation $N_n^2 = A_n + M_n$.

Solution:

(a) We have

$$\mathbb{E}(\Delta N_{i-1}^2 \,|\, \mathcal{F}_{i-1}) = \mathbb{E}(N_i^2 - N_{i-1}^2 \,|\, \mathcal{F}_{i-1}) \overset{(*)}{=} \mathbb{E}(N_i^2 \,|\, \mathcal{F}_{i-1}) - N_{i-1}^2$$

where in $(*)$ we used the linearity of conditional expectation and the fact that N_{i-1}^2 is $\mathcal{F}_{i-1}$-measurable.

$$\begin{aligned}\mathbb{E}((\Delta N_{i-1})^2 \,|\, \mathcal{F}_{i-1}) = \mathbb{E}((N_i - N_{i-1})^2 \,|\, \mathcal{F}_{i-1}) = \mathbb{E}(N_i^2 - 2N_iN_{i-1} + N_{i-1}^2 \,|\, \mathcal{F}_{i-1}) \overset{(**)}{=} \\ \mathbb{E}(N_i^2 \,|\, \mathcal{F}_{i-1}) - 2\mathbb{E}(N_i \,|\, \mathcal{F}_{i-1})N_{i-1} + N_{i-1}^2 \overset{(***)}{=} \\ \mathbb{E}(N_i^2 \,|\, \mathcal{F}_{i-1}) - 2N_{i-1}N_{i-1} + N_{i-1}^2 = \mathbb{E}(N_i^2 \,|\, \mathcal{F}_{i-1}) - N_{i-1}^2\end{aligned}$$

where in $(**)$ we used linearity, separation of known info and the fact that N_{i-1}^2 is $\mathcal{F}_{i-1}$-measurable; and in $(***)$ we used the martingale property.

(b) We have seen in class that A_n and M_n can be calculated with the following formulas:

$$A_n = \mathbb{E}(N_1^2) + \sum_{i=2}^{n} (\mathbb{E}(N_i^2 | \mathcal{F}_{i-1}) - N_{i-1}^2) \overset{\text{(a)}}{=} \mathbb{E}(N_1^2) + \sum_{i=2}^{n} \mathbb{E}((\Delta N_{i-1})^2 \,|\, \mathcal{F}_{i-1}),$$

$$M_n = \sum_{i=1}^{n} (N_i^2 - \mathbb{E}(N_i^2 | \mathcal{F}_{i-1})).$$

(c) Now (M_n) is a martingale with zero expectation, so $\mathbb{E}(M_n) = 0$.
By the decomposition we have $N_n^2 = A_n + M_n$, thus $\mathbb{E}(N_n^2) = \mathbb{E}(A_n)$. Now the Pythagorean theorem for square-integrable martingales follows from the above formula for A_n combined with the linearity of expectation and the tower property.

Remark: Why do we call this formula a „Pythagorean theorem"?

Pythagorean theorem: $a^2 + b^2 = c^2$.

More precisely: if $\underline{v}$ and $\underline{w}$ are orthogonal, then $\|\underline{v} + \underline{w}\|^2 = \|\underline{v}\|^2 + \|\underline{w}\|^2$.

Using slightly different notation: if $\langle \underline{v}, \underline{w} \rangle = 0$, then $\langle \underline{v} + \underline{w}, \underline{v} + \underline{w} \rangle = \langle \underline{v}, \underline{v} \rangle + \langle \underline{w}, \underline{w} \rangle$.

Now if (N_n) is a square integrable martingale, then $\mathbb{E}(N_1 \Delta N_1) = 0$, because

$$\mathbb{E}(N_1 \Delta N_1) = \mathbb{E}(\mathbb{E}(N_1 \Delta N_1 \,|\, \mathcal{F}_1)) = \mathbb{E}(N_1 \mathbb{E}(\Delta N_1 \,|\, \mathcal{F}_1)) = \mathbb{E}(N_1 \cdot 0) = 0.$$

Thus ΔN_1 is „orthogonal" to N_1. Form this we obtain:

$$\mathbb{E}(N_2^2) = \mathbb{E}((N_1 + \Delta N_1)^2) = \mathbb{E}(N_1^2) + \mathbb{E}((\Delta N_1)^2),$$

this is the $n = 2$ case of the Pythagorean theorem for square-integrable martingales.

3. Let X_i, $i = 1, 2, \ldots$ be i.i.d. with $\mathbb{P}(X_i = 1) = 1/3$ and $\mathbb{P}(X_i = -1) = 2/3$. Let $S_n := X_1 + X_2 + \cdots + X_n$. In other words, (S_n) is a biased random walk with downward drift.

   (a) For what values of $\lambda \in (0, +\infty)$ will $M_n := \lambda^{S_n}$ be a martingale?
   Hint: First calculate $\mathbb{E}(\lambda^{X_n})$, then calculate $\mathbb{E}(M_n \,|\, \mathcal{F}_{n-1})$.

   (b) For $a \in \mathbb{Z}$ let $T_a = \min\{\, n : S_n = a \,\}$. Use the optional stopping theorem for the above martingale M_n and the stopping time $\tau = T_{-a} \wedge T_b$ in order to calculate the probability $\mathbb{P}(T_{-a} > T_b)$ (i.e., the probability of the event that the walker hits b before hitting $-a$) where $a, b \geq 0$.

   (c) Calculate $\lim_{a\to\infty} \mathbb{P}(T_{-a} > T_b)$ for any $b = 0, 1, 2, \ldots$.

   (d) What is the probability that the walker ever hits level b? I.e., what is $\mathbb{P}(T_b < \infty)$, $b = 0, 1, 2, \ldots$?

   (e) Denote by $\mathcal{M} = \max_{n\in\mathbb{N}} S_n$. Show that $\mathcal{M}$ has geometric distribution and find its parameter.
   Hint: Observe that $\mathcal{M} \geq b$ if and only if $T_b < \infty$, $b = 0, 1, 2, \ldots$

   (f) Show that $\mathbb{P}(T_{-100} < \infty) = 1$. *Hint:* Repeat the argument used in (c) and (d).

Solution:

(a) Let $(\mathcal{F}_n)_{n=0}^{\infty}$ be the natural filtration of (X_n).

$$\mathbb{E}(M_n|\mathcal{F}_{n-1}) = \mathbb{E}(\lambda^{S_n}|\mathcal{F}_{n-1}) = \mathbb{E}(\lambda^{S_{n-1}}\lambda^{X_n}|\mathcal{F}_{n-1}) = \lambda^{S_{n-1}}\mathbb{E}(\lambda^{X_n}|\mathcal{F}_{n-1}) = M_{n-1}\mathbb{E}(\lambda^{X_n})$$

Thus we want $\mathbb{E}(\lambda^{X_n}) = 1$ and then (M_n) will be a martingale. Now $\mathbb{E}(\lambda^{X_n}) = \frac{1}{3}\lambda + \frac{2}{3}\frac{1}{\lambda}$ thus we want $\frac{1}{3}\lambda + \frac{2}{3}\frac{1}{\lambda} = 1$. Solving this quadratic equation for λ we find that $\lambda = 1$ or $\lambda = 2$. The first case is trivial ($M_n \equiv 1$), so from now on we assume that the second case holds: $M_n = 2^{S_n}$.

(b) Note that $\tau = T_{-a} \wedge T_b$ is a stopping time (it is the hitting time of the set $\{-a, b\}$) and that $\mathbb{P}(0 \leq M_{n\wedge\tau} \leq 2^b) = 1$, so the stopped martingale is bounded. The property $\mathbb{P}(\tau < \infty) = 1$ is intuitively obvious (the walker will sooner or later either hit $-a$ or b) and it is OK to omit the proof. So we can apply the optional stopping theorem to infer $\mathbb{E}(M_\tau) = \mathbb{E}(M_0)$. Now observe that if $T_{-a} > T_b$ then $\tau = T_b$ and thus $M_\tau = M_{T_b} = 2^{S_{T_b}} = 2^b$ and if $T_{-a} < T_b$ then $\tau = T_{-a}$ and thus $M_\tau = M_{T_{-a}} = 2^{S_{T_{-a}}} = 2^{-a}$. Moreover $\mathbb{P}(T_{-a} < T_b) = 1 - \mathbb{P}(T_{-a} > T_b)$, so

$$1 = 2^{S_0} = \mathbb{E}(M_0) = \mathbb{E}(M_\tau) = 2^{-a}(1 - \mathbb{P}(T_{-a} > T_b)) + 2^b\mathbb{P}(T_{-a} > T_b).$$

Rearranging this equation we obtain $\mathbb{P}(T_{-a} > T_b) = \frac{1-2^{-a}}{2^b-2^{-a}}$ and $\mathbb{P}(T_{-a} < T_b) = \frac{2^b-1}{2^b-2^{-a}}$.

(c) $\lim_{a\to\infty} \mathbb{P}(T_{-a} > T_b) = 2^{-b}$, $b = 0, 1, 2, \ldots$.

(d) We will show that $\mathbb{P}(T_b < \infty) = 2^{-b}$. Indeed, $a \leq a'$ implies $T_{-a} \leq T_{-a'}$, so we have $\{T_b < T_{-a}\} \subseteq \{T_b < T_{-a'}\}$. Also note that $T_{-a} \geq a$, since a walker starting from 0 has to perform at least a steps before reaching site $-a$. Thus we have $\lim_{a\to\infty} T_{-a} = \infty$, and thus

$$\lim_{a\to\infty} \{T_b < T_{-a}\} = \cup_{a=0}^{\infty}\{T_b < T_{-a}\} = \{T_b < \infty\}.$$

Thus $\lim_{a\to\infty} \mathbb{P}(T_b < T_{-a}) = \mathbb{P}(T_b < \infty)$ by the monotone convergence theorem. Thus the solution to (d) follows from the solution to (c).

(e) We will show that $\mathcal{M} \sim \mathrm{GEO}(\frac{1}{2})$. Observe that $\mathcal{M} \geq b$ if and only if $T_b < \infty$ for any $b \in \mathbb{N}$. Thus

$$\mathbb{P}(\mathcal{M} = b) = \mathbb{P}(\mathcal{M} \geq b) - \mathbb{P}(\mathcal{M} \geq b+1) = \mathbb{P}(T_b < \infty) - \mathbb{P}(T_{b+1} < \infty) = 2^{-b} - 2^{-(b+1)} = \frac{1}{2}(1 - \frac{1}{2})^b.$$

(f) If $b \leq b'$ then $\{T_{-100} < T_b\} \subseteq \{T_{-100} < T_{b'}\}$, thus $\lim_{b\to\infty}\{T_{-100} < T_b\} = \cup_{b=0}^{\infty}\{T_{-100} < T_b\} = \{T_{-100} < \infty\}$, thus $\mathbb{P}(T_{-100} < \infty) = \lim_{b\to\infty} \mathbb{P}(T_{-100} < T_b) = \lim_{b\to\infty} \frac{2^b-1}{2^b-2^{-100}} = 1$.

4. Let X_i, $i = 1, 2, \ldots$ be i.i.d. with $\mathbb{P}(X_i = 1) = \frac{2}{3}$, $\mathbb{P}(X_i = -1) = \frac{1}{3}$. Let $S_n := X_1 + X_2 + \cdots + X_n$.

   Thus (S_n) is a biased random walk with an upward drift. Let $(\mathcal{F}_n)_{n\geq 0}$ denote the filtration generated by the process (S_n).

   (a) Find the discrete Doob-Meyer decomposition of the process (S_n), i.e., write $S_n = A_n + M_n$, where (A_n) is a predictable process and (M_n) is a martingale with zero expectation. Explicitly state the simplest possible formula for A_n.

   (b) Denote by $\tau = \min\{\, n \,:\, S_n = 100 \,\}$ the first time when the walker reaches level 100.
   Use part (a) and the optional stopping theorem to calculate $\mathbb{E}[\tau]$.
   Instruction: You don't have to fully verify that the optional stopping theorem can be applied here. In particular, don't prove that $\lim_{n\to\infty} \mathbb{E}(M_{n\wedge\tau}) = \mathbb{E}(M_\tau)$. But why do we have $\mathbb{P}(\tau < +\infty) = 1$?

Solution:

(a) $\Delta S_n = S_{n+1} - S_n = X_{n+1}$. $\Delta A_n = \mathbb{E}[\Delta S_n \,|\, \mathcal{F}_n] = \mathbb{E}[X_{n+1} \,|\, \mathcal{F}_n] = \mathbb{E}[X_{n+1}] = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$.
$S_0 = 0$, thus $A_n = \sum_{i=0}^{n-1} \Delta A_i = \frac{n}{3}$ and $M_n = S_n - \frac{n}{3}$ is a martingale with $M_0 = 0$.

(b) $0 = M_0 = \mathbb{E}[M_0] = \mathbb{E}[M_\tau] = \mathbb{E}[S_\tau - \frac{\tau}{3}] = 100 - \mathbb{E}[\tau]/3$, thus $\mathbb{E}[\tau] = 300$.
$\mathbb{P}(\tau < +\infty) = 1$ holds by the same argument as in sub-exercise (f) of exercise 3 above.