Stoch. Anal. HW assignment 3. Solution

Note: Each of the 4 questions is worth 10 marks. Write the solutions of different exercises on different pages.

1. Let (B_t) and (B_t^*) denote independent standard Brownian motions.

   (a) *Time inversion:* Let $\widetilde{B}_t := t \cdot B_{1/t}$ if $t > 0$, $\widetilde{B}_0 := 0$. Show that $(\widetilde{B}_t)$ is a standard Brownian motion.

   (b) *Time reversal:* Let $X_t := B(1) - B(1-t)$ for any $0 \le t \le 1$. Show that $(X_t)_{0 \le t \le 1}$ is also a standard Brownian motion on the time interval $[0, 1]$.

   (c) *Superposition:* Let $\widehat{B}_t = \frac{1}{\sqrt{2}} (B_t + B_t^*)$. Show that $(\widehat{B}_t)$ is also a standard Brownian motion.

   Hint: We have already learnt two equivalent definitions of standard Brownian motion: in each of the sub-exercises (a), (b) and (c), use the definition that is most convenient.

   Solution:

   (a) $\mathbb{E}(\widetilde{B}_t) = \mathbb{E}(t \cdot B_{1/t}) = t \cdot \mathbb{E}(B_{1/t}) = 0$.

   $$\mathrm{Cov}(\widetilde{B}_s, \widetilde{B}_t) = \mathrm{Cov}(sB_{1/s}, tB_{1/t}) = st\mathrm{Cov}(B_{1/s}, B_{1/t}) = st \min\left\{\frac{1}{s}, \frac{1}{t}\right\} = \frac{st}{\max\{s,t\}} = \min\{s,t\}.$$

   Also note that $(\widetilde{B}_t)$ is a Gaussian process, as we now explain. First note that we know from class that (B_t) is a Gaussian process, hence $(B(1/t_1), \ldots, B(1/t_n))$ has multivariate normal distribution. From this it follows that $(t_1 B(1/t_1), \ldots, t_n B(1/t_n))$ also has multivariate normal distribution, since the linear transformation of a multivariate normal random vector is also a multivariate normal random vector (see Lecture 5). Hence $(\widetilde{B}_t)$ is a Gaussian process.

   We know that $t \mapsto B_t$ is a continuous function, thus $t \mapsto \widetilde{B}_t$ is also continuous on $(0, +\infty)$. It remains to show that $\lim_{t \to 0_+} \widetilde{B}_t = 0$. Now (B_t) and $(\widetilde{B}_t)$ are both continuous Gaussian processes on $(0, +\infty)$ with the same mean and covariance structure, hence their law agrees on $(0, +\infty)$. This means that the restriction of $(\widetilde{B}_t)$ to $(0, +\infty)$ behaves the same way as the restriction of (B_t) to $(0, +\infty)$. We know that $\mathbb{P}(\lim_{t \to 0_+} B_t = 0) = 1$, thus $\mathbb{P}(\lim_{t \to 0_+} \widetilde{B}_t = 0) = 1$. This shows that $(\widetilde{B}_t)$ is continuous on $[0, +\infty)$. This shows that $(\widetilde{B}_t)$ is a Brownian motion, according to the „Gaussian" characterization of B.M. (see Lecture 7).

   (b) Let $0 \le t \le 1$. $\mathbb{E}(X_t) = \mathbb{E}(B(1) - B(1-t)) = 0 - 0 = 0$.
   Let $s, t \in [0, 1]$. Assume $s \le t$. We want to show $\mathrm{Cov}(X_s, X_t) = s$.

   $$\mathrm{Cov}(X_s, X_t) = \mathrm{Cov}(B(1) - B(1-s), B(1) - B(1-t)) =$$
   $$\mathrm{Cov}(B(1), B(1)) - \mathrm{Cov}(B(1), B(1-s)) - \mathrm{Cov}(B(1), B(1-t)) + \mathrm{Cov}(B(1-s), B(1-t)) =$$
   $$1 \wedge 1 - 1 \wedge (1-s) - 1 \wedge (1-t) + (1-s) \wedge (1-t) = 1 - (1-s) - (1-t) + (1-t) = s + t - t = s.$$

   Also note that (X_t) is a Gaussian process, since it arises from another Gaussian process (Brownian motion) using linear operations.
   Also note that $(X_t)_{0 \le t \le 1}$ is continuous. Hence $(X_t)_{0 \le t \le 1}$ is Brownian motion.

   (c) $(\widehat{B}_t)$ arises using linear operations from two independent Gaussian processes, hence it is also a Gaussian process.
   $(\widehat{B}_t)$ is the sum of two continuous functions, hence it is continuous.
   $\mathbb{E}(\widehat{B}_t) = \mathbb{E}(\frac{1}{\sqrt{2}} (B_t + B_t^*)) = \frac{1}{\sqrt{2}} (\mathbb{E}(B_t) + \mathbb{E}(B_t^*)) = \frac{1}{\sqrt{2}} (0 + 0) = 0$.

   $$\mathrm{Cov}(\widehat{B}_s, \widehat{B}_t) = \mathrm{Cov}(\frac{1}{\sqrt{2}} (B_s + B_s^*), \frac{1}{\sqrt{2}} (B_t + B_t^*)) = \frac{1}{2}\mathrm{Cov}(B_s + B_s^*, B_t + B_t^*) =$$
   $$\frac{1}{2}\left(\mathrm{Cov}(B_s, B_t) + \frac{1}{2}\mathrm{Cov}(B_s, B_t^*) + \frac{1}{2}\mathrm{Cov}(B_s^*, B_t) + \frac{1}{2}\mathrm{Cov}(B_s^*, B_t^*)\right) =$$
   $$\frac{1}{2}(s \wedge t + 0 + 0 + s \wedge t) = s \wedge t.$$

   Hence $(\widehat{B}_t)$ is a Brownian motion.

2. Let (B_t) denote standard Brownian motion. Let $0 \leq t_1 < t_2 < t_3$.

   (a) Write down the covariance matrix $\underline{\underline{C}}$ of the random vector $(B_{t_1}, B_{t_2}, B_{t_3})^T$.

   (b) Let Y be a standard normal r.v. independent from (B_t). Let $Z := \frac{t_3-t_2}{t_3-t_1} B_{t_1} + \frac{t_2-t_1}{t_3-t_1} B_{t_3} + aY$.
   In words: we obtain Z by linearly interpolating between B_{t_1} and B_{t_3} and then adding some independent noise aY. How to choose the constant $a > 0$ if we want $(B_{t_1}, Z, B_{t_3})^T$ to have the same joint distribution as $(B_{t_1}, B_{t_2}, B_{t_3})^T$?

   (c) What is the conditional density function of B_{t_2} given $B_{t_1} = x$ and $B_{t_3} = y$ for some $x, y \in \mathbb{R}$?

Solution:

(a) $\underline{\underline{C}} = \begin{pmatrix} t_1 & t_1 & t_1 \\ t_1 & t_2 & t_2 \\ t_1 & t_2 & t_3 \end{pmatrix}$, in other words: $\underline{\underline{C}} = (c_{i,j})_{i,j=1}^3$, where $c_{i,j} = \mathrm{Cov}(B(t_i), B(t_j)) = t_i \wedge t_j$.

(b) We know from class that the random vector $(B_{t_1}, B_{t_2}, B_{t_3})^T$ has multivariate normal distribution. Now $(B_{t_1}, Z, B_{t_3})^T$ also has multivariate normal distribution, because it can be obtained from $(B_{t_1}, B_{t_2}, B_{t_3})^T$ and an independent standard normal Y using linear operations. So it suffices to choose $a > 0$ such that $(B_{t_1}, B_{t_2}, B_{t_3})^T$ and $(B_{t_1}, Z, B_{t_3})^T$ have the same mean vector and covariance matrix (since this data already determines their joint distribution, c.f. Lecture 5). For any value of the parameter a we have

$$\mathbb{E}(Z) = \frac{t_3 - t_2}{t_3 - t_1} 0 + \frac{t_2 - t_1}{t_3 - t_1} 0 + a0 = 0 = \mathbb{E}(B_{t_2}),$$

so we have to choose a in a way that the following equations all hold:

$$\mathrm{Cov}(Z, B_{t_1}) = \mathrm{Cov}(B_{t_2}, B_{t_1}), \quad \mathrm{Cov}(Z, Z) = \mathrm{Cov}(B_{t_2}, B_{t_2}), \quad \mathrm{Cov}(Z, B_{t_3}) = \mathrm{Cov}(B_{t_2}, B_{t_3}).$$

We can rewrite the definition of Z as

$$Z = B_{t_1} + \frac{t_2 - t_1}{t_3 - t_1}(B_{t_3} - B_{t_1}) + aY.$$

Note that B_{t_1}, $B_{t_3} - B_{t_1}$ and Y are independent, so we have

$$\mathrm{Cov}(Z, B_{t_1}) = \mathrm{Cov}(B_{t_1}, B_{t_1}) + 0 + 0 = t_1 = \mathrm{Cov}(B_{t_2}, B_{t_1}),$$

for any value of $a \in \mathbb{R}$. Also,

$$\mathrm{Cov}(Z, B_{t_3}) = \mathrm{Cov}(Z, B_{t_1} + (B_{t_3} - B_{t_1})) = \mathrm{Cov}(Z, B_{t_1}) + \mathrm{Cov}(Z, B_{t_3} - B_{t_1}) =$$
$$t_1 + 0 + \mathrm{Cov}\left(\frac{t_2 - t_1}{t_3 - t_1}(B_{t_3} - B_{t_1}), B_{t_3} - B_{t_1}\right) + 0 = t_1 + \frac{t_2 - t_1}{t_3 - t_1}(t_3 - t_1) = t_2 = \mathrm{Cov}(B_{t_2}, B_{t_3}),$$

for any value of $a \in \mathbb{R}$. So we have to choose $a > 0$ such that $\mathrm{Cov}(Z, Z) = \mathrm{Cov}(B_{t_2}, B_{t_2}) = t_2$ holds.

$$\mathrm{Cov}(Z, Z) = \mathrm{Var}(Z) = \mathrm{Var}(B_{t_1}) + \mathrm{Var}\left(\frac{t_2 - t_1}{t_3 - t_1}(B_{t_3} - B_{t_1})\right) + \mathrm{Var}(aY) = t_1 + \frac{(t_2 - t_1)^2}{t_3 - t_1} + a^2.$$

We want this to be equal to t_2, thus we want

$$a^2 = \frac{(t_2 - t_1)(t_3 - t_2)}{t_3 - t_1}, \qquad a = \sqrt{\frac{(t_2 - t_1)(t_3 - t_2)}{t_3 - t_1}}.$$

(c) By (b), the conditional distribution of B_{t_2} given $B_{t_1} = x$ and $B_{t_3} = y$ is the the same as the conditional distribution of Z given $B_{t_1} = x$ and $B_{t_3} = y$.
Under this conditioning, we have $Z = \frac{t_3-t_2}{t_3-t_1} x + \frac{t_2-t_1}{t_3-t_1} y + aY$, so Z has normal distribution $\mathcal{N}(\mu, \sigma^2)$ with mean $\mu = \frac{t_3-t_2}{t_3-t_1} x + \frac{t_2-t_1}{t_3-t_1} y$ and variance $\sigma^2 = a^2 = \frac{(t_2-t_1)(t_3-t_2)}{t_3-t_1}$. Thus the conditional density function of B_{t_2} under this conditioning is $f(z) = \frac{1}{a\sqrt{2\pi}} \exp\left(-\frac{(z-\mu)^2}{2a^2}\right)$.

3. *Quadratic variation of Brownian motion.* Let (B_t) denote a standard Brownian motion. Given the partition $\Delta_n = \{t_0, t_1, \ldots, t_n\}$, $t_0 < t_1 < \cdots < t_n$, $t_0 = 0, t_n = t$, let $Q_n := \sum_{k=1}^n \big(B(t_k) - B(t_{k-1})\big)^2$.

   (a) Calculate $\mathbb{E}(Q_n)$ and $\mathrm{Var}(Q_n)$.
   Hint: You can find an important ingredient of this calculation in HW1.2(d)!

   (b) Calculate the quadratic variation of (B_t). More precisely, show that $[B]_t = t$.
   Hint: You have to show that $Q_n \xrightarrow{\mathbb{P}} t$ as $|\Delta_n| \to 0$. using part (a) and *Chebyshev's inequality.*

Solution:

(a) The increment $B(t_k) - B(t_{k-1})$ has $\mathcal{N}(0, \sigma^2)$ distribution with $\sigma^2 = t_k - t_{k-1}$.
We have $\mathbb{E}[(B(t_k) - B(t_{k-1}))^2] = \mathrm{Var}[B(t_k) - B(t_{k-1})] = t_k - t_{k-1}$ and by the linearity of expectation we obtain $\mathbb{E}(Q_n) = \sum_{k=1}^n (t_k - t_{k-1}) = t$.
The increments of (B_t) are independent, thus the variance of the sum is equal to the sum of the variances: $\mathrm{Var}(Q_n) = \sum_{k=1}^n \mathrm{Var}\left((B(t_k) - B(t_{k-1}))^2\right) = \sum_{k=1}^n 2(t_k - t_{k-1})^2$ where the last equality holds by Homework 1.2(d).

(b)
$$\mathbb{P}[|Q_n - t| \geq \varepsilon] = \mathbb{P}[|Q_n - \mathbb{E}(Q_n)| \geq \varepsilon] \leq \frac{\mathrm{Var}(Q_n)}{\varepsilon^2} = \frac{\sum_{k=1}^n 2(t_k - t_{k-1})^2}{\varepsilon^2}$$
Now $\sum_{k=1}^n (t_k - t_{k-1})^2 \leq \sum_{k=1}^n |\Delta_n|(t_k - t_{k-1}) = |\Delta_n| t \to 0$ as $n \to \infty$.
Thus for every $\varepsilon > 0$ we have $\lim_{|\Delta_n| \to 0} \mathbb{P}[|Q_n - t| \geq \varepsilon] = 0$, i.e., $Q_n \xrightarrow{\mathbb{P}} t$ as $|\Delta_n| \to 0$.

Remark:

Let (B_t) and (B_t^*) denote independent standard Brownian motions.

Let $T_n = \sum_{k=1}^n \big(B(t_k) - B(t_{k-1})\big) \cdot \big(B^*(t_k) - B^*(t_{k-1})\big)$. Let us also calculate the mutual variation of (B_t) and (B_t^*). More precisely, let us show that $[B, B^*]_t = 0$. According to Lecture 8, we have to show that $T_n \xrightarrow{\mathbb{P}} 0$ as $|\Delta_n| \to 0$. We have calculated $[B, B^*]_t = 0$ using polarization in class (see Lecture 8), but that calculation was mysterious. Here we present a direct proof. We have

$$\mathbb{E}\left[\big(B(t_k) - B(t_{k-1})\big)\big(B^*(t_k) - B^*(t_{k-1})\big)\right] \overset{(*)}{=} \mathbb{E}\left[\big(B(t_k) - B(t_{k-1})\big)\right] \mathbb{E}\left[\big(B^*(t_k) - B^*(t_{k-1})\big)\right] = 0 \cdot 0 = 0,$$

where $(*)$ holds because (B_t) and (B_t^*) are independent. Thus $\mathbb{E}[T_n] = 0$ by linearity.

We have

$$\mathrm{Var}\left[\big(B(t_k) - B(t_{k-1})\big)\big(B^*(t_k) - B^*(t_{k-1})\big)\right] = \mathbb{E}\left[\big(B(t_k) - B(t_{k-1})\big)^2\big(B^*(t_k) - B^*(t_{k-1})\big)^2\right] - 0^2 = \mathbb{E}\left[\big(B(t_k) - B(t_{k-1})\big)^2\right] \mathbb{E}\left[\big(B^*(t_k) - B^*(t_{k-1})\big)^2\right] = (t_k - t_{k-1})^2.$$

Thus, by independence of increments of both (B_t) and (B_t^*) as well as the independence of (B_t) and (B_t^*), we obtain $\mathrm{Var}(T_n) = \sum_{k=1}^n (t_k - t_{k-1})^2$.

$\mathbb{P}[|T_n - 0| \geq \varepsilon] = \mathbb{P}[|T_n - \mathbb{E}(T_n)| \geq \varepsilon] \leq \frac{\mathrm{Var}(T_n)}{\varepsilon^2} = \frac{\sum_{k=1}^n (t_k - t_{k-1})^2}{\varepsilon^2}$

Thus for every $\varepsilon > 0$ we have $\lim_{|\Delta_n| \to 0} \mathbb{P}[|T_n - 0| \geq \varepsilon] = 0$, i.e., $T_n \xrightarrow{\mathbb{P}} 0$ as $|\Delta_n| \to 0$.

4. *The problem with stochastic integrals.* Recall the definition of the Stieltjes integral. The goal of this exercise is to „naively" try to calculate the integral

$$\int_0^t B(s)\,\mathrm{d}B(s),$$

where (B_t) is standard Brownian motion. We will then soon realize that we get into trouble with our naive approach and that stochastic integrals behave differently from Stieltjes integrals.

We consider a sequence of partitions (Δ_n) of $[0,t]$ satisfying $|\Delta_n| \to 0$. Let us define

$$L_n = \sum_{k=1}^{n} B(t_{k-1}) \cdot (B(t_k) - B(t_{k-1})), \qquad R_n = \sum_{k=1}^{n} B(t_k) \cdot (B(t_k) - B(t_{k-1})).$$

In the case of L_n we chose the sample point t_k^* to be the left endpoint of the interval $[t_{k-1}, t_k]$, in the case of R_n we chose t_k^* to be the right endpoint. If this was a proper Stieltjes integral then we would have

$$\int_0^t B(s)\,\mathrm{d}B(s) = \lim_{|\Delta_n|\to 0} L_n = \lim_{|\Delta_n|\to 0} R_n.$$

However, the truth turns out to be different:

(a) Show that $R_n - L_n$ converges in probability as $|\Delta_n| \to 0$, but the limit is non-zero.

(b) Find the limit of L_n (in probability) as $|\Delta_n| \to 0$.
Hint: First calculate the limit of $L_n + R_n$ as $n \to \infty$.

(c) For comparison, let us assume that $b : \mathbb{R}_+ \to \mathbb{R}$ is a continuously differentiable function satisfying $b(0) = 0$. Find the value of the Stieltjes integral $\int_0^t b(s)\,\mathrm{d}b(s)$.

Solution:

(a) $R_n - L_n = Q_n$ from the previous exercise of this homework sheet. Thus $R_n - L_n \xrightarrow{\mathbb{P}} t$.

(b) Observe that for any partition Δ_n we have

$$R_n + L_n = \sum_{k=1}^{n} (B(t_k) + B(t_{k-1})) \cdot (B(t_k) - B(t_{k-1})) =$$
$$\sum_{k=1}^{n} (B^2(t_k) - B^2(t_{k-1})) = B^2(t_n) - B^2(t_0) = B^2(t) - B^2(0) = B^2(t)$$

Therefore we obtain

$$L_n = \frac{(R_n + L_n) - (R_n - L_n)}{2} = \frac{B^2(t) - Q_n}{2} \xrightarrow{\mathbb{P}} \frac{1}{2}\left(B^2(t) - t\right)$$

(c) Let us use the formula relating Stieltjes integrals to Riemann integrals in the case when we integrate w.r.t. a continuously differentiable function (see Lecture 8):

$$\int_0^t b(s)\,\mathrm{d}b(s) = \int_0^t b(s)b'(s)\,\mathrm{d}s = \left[\frac{1}{2}b^2(s)\right]_0^t = \frac{1}{2}b^2(t).$$

Alternatively, in this case $Q_n \to 0$ (the quadratic variation of a continuous function with finite total variation vanishes, see Lecture 8), thus $R_n - L_n = Q_n \to 0$ and $L_n = \frac{b^2(t) - Q_n}{2} \to \frac{1}{2}b^2(t)$.