Stoch. Anal. HW assignment 4. Solutions

1. *Laplace transform of the distribution of hitting time of Brownian motion with upward drift.*

   Let (B_t) denote the standard Brownian motion. Let $0 \le \mu < +\infty$. Let us define $X_t := B_t + \mu t$. We call (X_t) the Brownian motion with upward drift μ. Let $\tau = \min\{\, t \;:\; X_t = 1 \,\}$ the hitting time of level 1. You may assume without proof that $\mathbb{P}(\tau < +\infty) = 1$.

   (a) For any $\alpha \in \mathbb{R}$ find a constant $\beta \in \mathbb{R}$ such that (M_t) is a martingale, where $M_t := \exp(\alpha X_t - \beta t)$. *Hint:* For martingales related to Brownian motion, see Lecture 7.

   (b) Apply the optional stopping theorem to calculate $\mathbb{E}\left(e^{-\lambda\tau}\right)$ for any $\lambda \in \mathbb{R}_+$.
   Hint: You can use the following continuous-time form of the optional stopping theorem: if (M_t) is a martingale, τ is a stopping time satisfying $\mathbb{P}(\tau < +\infty) = 1$ and if there exists a constant $C \in \mathbb{R}_+$ such that $\mathbb{P}(|M_{t\wedge\tau}| \le C) = 1$ for any $t \ge 0$ then we have $\mathbb{E}(M_\tau) = \mathbb{E}(M_0)$.

   Solution:

   (a) $M_t = \exp(\alpha(B_t + \mu t) - \beta t) = \exp(\alpha B_t - (\beta - \alpha\mu)t)$. We know from Lecture 7 that $\left(\exp\left(\alpha B_t - \frac{1}{2}\alpha^2 t\right)\right)$ is a martingale, so we want to choose β so that we have $\beta - \alpha\mu = \frac{1}{2}\alpha^2$. So we have to choose

   $$\beta = \beta(\alpha) = \frac{1}{2}\alpha^2 + \alpha\mu.$$

   (b) Given $\lambda > 0$ we want to choose α so that $\beta(\alpha) = \lambda$. We want to find α such that $\frac{1}{2}\alpha^2 + \alpha\mu = \lambda$. We want $\alpha^2 + 2\mu\alpha - 2\lambda = 0$. The solutions of this quadratic equation are

   $$\alpha_{1,2} = \frac{-2\mu \pm \sqrt{4\mu^2 + 8\lambda}}{2} = -\mu \pm \sqrt{\mu^2 + 2\lambda}$$

   Note that $\alpha_1 > 0$ and $\alpha_2 < 0$. We will have to use α_1 for reasons that will become apparent later.
   So if we define $M_t = \exp(\alpha_1 X_t - \beta(\alpha_1)t) = \exp(\alpha_1 X_t - \lambda t)$ then (M_t) is a martingale.
   Certainly $0 \le M_t$ for all t and if $t \le \tau$ then $X_t \le 1$, so for all $t \in \mathbb{R}_+$ we have $X_{t\wedge\tau} \le 1$. Thus

   $$M_{t\wedge\tau} = \exp(\alpha_1 X_{t\wedge\tau} - \lambda(t\wedge\tau)) \le \exp(\alpha_1 X_{t\wedge\tau}) \overset{(*)}{\le} \exp(\alpha_1).$$

   We used $\alpha_1 > 0$ in the inequality marked by $(*)$. This is why we have to use α_1 and not α_2.
   We have shown that $\mathbb{P}(|M_{t\wedge\tau}| \le e^{\alpha_1}) = 1$, so we can use the optional stopping theorem to conclude

   $$1 = M_0 = \mathbb{E}[M_0] = \mathbb{E}[M_\tau] = \mathbb{E}\left[\exp(\alpha_1 X_\tau - \lambda\tau)\right] \overset{(**)}{=} \mathbb{E}\left[\exp(\alpha_1 - \lambda\tau)\right] = e^{\alpha_1}\mathbb{E}\left(e^{-\lambda\tau}\right),$$

   where in the equation marked by $(**)$ we used that $X_\tau = 1$. Thus

   $$\mathbb{E}\left(e^{-\lambda\tau}\right) = e^{-\alpha_1} = \exp\left(\mu - \sqrt{\mu^2 + 2\lambda}\right).$$

   Note that clearly there would have been trouble if we tried to use α_2 instead of α_1, because we would have obtained $\mathbb{E}\left(e^{-\lambda\tau}\right) = e^{-\alpha_2} = \exp\left(\mu + \sqrt{\mu^2 + 2\lambda}\right) > 1$, which contradicts $\mathbb{P}(0 \le e^{-\lambda\tau} \le 1) = 1$.

Remark. Let us consider the $\mu = 0$ case (thus $X_t \equiv B_t$). We have learnt in class that in this case τ has *Lévy distribution* and the density function $t \mapsto f(t)$ of τ was calculated on lecture 10. We have just calculated the Laplace transform of f:

$$\int_0^\infty e^{-\lambda t} f(t)\,\mathrm{d}t = \exp\left(-\sqrt{2\lambda}\right).$$

This equality holds, since both sides are equal to $\mathbb{E}\left(e^{-\lambda\tau}\right)$.

Note that for any $\lambda > 0$ we have $\mathbb{E}\left(e^{\lambda\tau}\right) = \int_0^\infty e^{\lambda t} f(t)\,\mathrm{d}t = +\infty$.

You will learn (or maybe have already learnt) more about the Lévy distribution on the *Limit Theorems of Probability* class (e.g., that the Lévy distribution is a *stable distribution*).

2. Let $S_n = \xi_1 + \cdots + \xi_n$, where $\xi_1, \xi_2, \ldots,$ are i.i.d. and $\mathbb{P}(\xi_k = 1) = \mathbb{P}(\xi_k = -1) = \frac{1}{2}$, $k \geq 1$. Let $(\mathcal{F}_n)_{n\geq 0}$ denote the natural filtration of the simple random walk (S_n), i.e., $\mathcal{F}_n = \sigma(S_1, \ldots, S_n) = \sigma(\xi_1, \ldots, \xi_n)$.

   (a) Show that any martingale (M_n) with $M_0 = 0$ adapted to $(\mathcal{F}_n)_{n\geq 0}$ is a discrete stochastic integral $(H \cdot S)_n$ of a predictable process (H_n) with respect to the martingale (S_n). Explicitly state the formula for H_n. *Hint:* Since M_n is $\sigma(\xi_1, \ldots, \xi_n)$-measurable, there exists a function $\varphi_n : \{-1, 1\}^n \to \mathbb{R}$ such that $M_n = \varphi_n(\xi_1, \ldots, \xi_n)$. Please use this notation in your proof. Also note that an earlier homework was a special case of this exercise.

   (b) Show that if X is an $\mathcal{F}_{100}$-measurable random variable then $X = \mathbb{E}[X] + \sum_{k=1}^{100} Y_k \cdot (S_k - S_{k-1})$ for some random variables $Y_1, \ldots, Y_{100}$, where Y_k is $\mathcal{F}_{k-1}$-measurable, $k = 1, \ldots, 100$.
   Hint: How do you create a martingale $(M_n)_{n=0}^{100}$ out of the random variable X?

Solution:

(a) We want $M_n = \sum_{i=1}^n H_i \cdot (S_i - S_{i-1}) = \sum_{i=1}^n H_i \xi_i$. Thus we want $\Delta M_{n-1} = M_n - M_{n-1} = H_n \xi_n$ for any $n \geq 0$. And as soon as we will find a predictable process (H_n) for which we have $M_i - M_{i-1} = H_i \xi_i$ for any $i \in \mathbb{N}$, we will obtain that $M_n = M_n - M_0 = \sum_{i=1}^n (M_i - M_{i-1}) = \sum_{i=1}^n H_i \xi_i = (H \cdot S)_n$, as desired.
Since (M_n) is a martingale we have $\mathbb{E}(M_i | \mathcal{F}_{i-1}) = M_{i-1}$. We observe that

$$M_{i-1} = \mathbb{E}(\varphi_i(\xi_1, \ldots, \xi_i) | \mathcal{F}_{i-1}) = \frac{\varphi_i(\xi_1, \ldots, \xi_{i-1}, 1) + \varphi_i(\xi_1, \ldots, \xi_{i-1}, -1)}{2}.$$

Since the value of ξ_i is either $+1$ or -1, in both of these cases we have

$$\Delta M_{i-1} = M_i - M_{i-1} = \varphi_i(\xi_1, \ldots, \xi_i) - \frac{\varphi_i(\xi_1, \ldots, \xi_{i-1}, 1) + \varphi_i(\xi_1, \ldots, \xi_{i-1}, -1)}{2} = \frac{\varphi_i(\xi_1, \ldots, \xi_{i-1}, 1) - \varphi_i(\xi_1, \ldots, \xi_{i-1}, -1)}{2} \xi_i.$$

Let us define $H_i := \frac{\varphi_i(\xi_1, \ldots, \xi_{i-1}, 1) - \varphi_i(\xi_1, \ldots, \xi_{i-1}, -1)}{2}$. This is indeed $\mathcal{F}_{i-1}$-measurable, so (H_i) is predictable and $M_i - M_{i-1} = H_i \xi_i$, as desired.

(b) We let $Z = X - \mathbb{E}[X]$ and $M_n = \mathbb{E}[\, Z \,|\, \mathcal{F}_n \,]$. This is the *Doob martingale* construction (c.f. lecture 7). Indeed, $(M_n)_{n=0}^{100}$ is a martingale by the tower property:

$$\mathbb{E}[M_n \,|\, \mathcal{F}_{n-1}] = \mathbb{E}[\, \mathbb{E}[\, Z \,|\, \mathcal{F}_n \,] \,|\, \mathcal{F}_{n-1}] = \mathbb{E}[\, Z \,|\, \mathcal{F}_{n-1} \,] = M_{n-1}, \qquad 1 \leq n \leq 100.$$

Also note that $M_0 = \mathbb{E}[Z \,|\, \mathcal{F}_0] = \mathbb{E}[Z] = 0$. We can thus apply part (a) of the exercise to obtain that $Z = M_{100} = (H \cdot S)_{100}$ for some $H_1, \ldots, H_{100}$, where H_k is $\mathcal{F}_{k-1}$-measurable, so

$$X = \mathbb{E}[X] + Z = \mathbb{E}[X] + \sum_{k=1}^{100} H_k \cdot (S_k - S_{k-1}).$$

Remark. Note that the above described *martingale representation property* of simple random walk is quite special: e.g. *lazy random walk* does not have this property. However, Brownian motion does possess this magical property: part (b) of this exercise was the discrete version of the famous *martingale representation theorem* which states that a square integrable random variable X that is measurable with respect to $\mathcal{F}_T$ for some $T \geq 0$ (where $(\mathcal{F}_t)$ is the filtration generated by a Brownian motion) can be written in terms of an Itô integral with respect to this Brownian motion. In other words, if $\mathbb{E}[X^2] < \infty$ and $X \in \mathcal{F}_T$, then

$$X = \mathbb{E}[X] + \int_0^T Y_s \, \mathrm{d}B_s,$$

for some $(\mathcal{F}_t)$-adapted process $(Y_t)_{0\leq t\leq T}$ in $L_2(\Omega \times [0, T])$. The „Brownian" analogue of part (a) states that every square integrable martingale (M_t) adapted to $(\mathcal{F}_t)$ is an Itô integral: $M_t = \int_0^t Y_s \, \mathrm{d}B_s$.

You will see the proof of the Brownian martingale representation theorem in a few weeks and you will see some applications to financial math in your *Financial Processes* course.

3. Recall the def. of stopping times (Lecture 4) and the σ-field $\mathcal{F}_\tau$ at the stopping time τ (Lecture 10).

   (a) Show that the collection of sets $\mathcal{F}_\tau$ is indeed a sigma-field. *Hint:* We defined σ-fields on Lecture 1.

   (b) Assume that we are in „discrete time" and define

   $$\mathcal{F}'_\tau = \{A \in \mathcal{F} \,:\, A \cap \{\tau = n\} \in \mathcal{F}_n,\ n \in \mathbb{N}\}.$$

   Show that $\mathcal{F}'_\tau = \mathcal{F}_\tau$.

   (c) Let (X_n) be a discrete-time stochastic process, let $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$ and let τ be a stopping time. Prove that X_τ is $\mathcal{F}_\tau$-measurable, i.e., that for any Borel set $B \in \mathcal{B}(\mathbb{R})$ we have $\{X_\tau \in B\} \in \mathcal{F}_\tau$.

   (d) Show that a discrete time Markov process (X_n) also has the strong Markov property.
   Hint: The plain Markov property is defined on Lecture 8 and the strong Markov property is defined on Lecture 10. You will have to show that $\mathbb{E}(f(X_{\tau+s}) \,|\, \sigma(X_\tau))$ satisfies the implicit definition (see Lecture 2) of $\mathbb{E}(f(X_{\tau+s}) \,|\, \mathcal{F}_\tau)$. The proof will involve an arbitrary event $A \in \mathcal{F}_\tau$, and here is a useful identity: $\mathbb{1}[A] = \sum_{n=0}^{\infty} \mathbb{1}[A \cap \{\tau = n\}]$.

Solution.

(a) It is trivial that $\emptyset \in \mathcal{F}_\tau$. Now we show that $\Omega \in \mathcal{F}_\tau$. Indeed: $\{\tau \le t\} \in \mathcal{F}_t$ since τ is a stopping time, therefore $\Omega \cap \{\tau \le t\} \in \mathcal{F}_t$ for all $t \ge 0$, thus $\Omega \in \mathcal{F}_\tau$.
Next, we show that if $A \in \mathcal{F}_\tau$ then the complement $A^c = \Omega \setminus A$ is in $\mathcal{F}_\tau$. We only need to check that $A^c \cap \{\tau \le t\} \in \mathcal{F}_t$ for all $t \ge 0$. This is indeed true, because

$$A^c \cap \{\tau \le t\} = \{\tau \le t\} \cap (A \cap \{\tau \le t\})^c,$$

and $\{\tau \le t\} \in \mathcal{F}_t$ (because τ is a stopping time), moreover $A \cap \{\tau \le t\} \in \mathcal{F}_t$ (because $A \in \mathcal{F}_\tau$), and because $\mathcal{F}_t$ is closed under taking complements and intersections.
The last thing we have to prove is that if $A_i \in \mathcal{F}_\tau$ for $i \in \mathbb{N}$, then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}_\tau$. Since $A_i \in \mathcal{F}_\tau$ for $i \in \mathbb{N}$, $A_i \cap \{\tau \le t\} \in \mathcal{F}_t$ for $t \ge 0, i \in \mathbb{N}$. Now since $\mathcal{F}_t$ is a sigma-field for all $t \ge 0$, $\cup_{i=1}^{n}(A_i \cap \{\tau \le t\}) \in \mathcal{F}_t$ for $t \ge 0$. But this is exactly $(\cup_{i=1}^{n} A_i) \cap \{\tau \le t\} \in \mathcal{F}_t$ for $t \ge 0$, so $\cup_{i=1}^{\infty} A_i \in \mathcal{F}_\tau$.

(b) We defined

$$\mathcal{F}'_\tau = \{A \in \mathcal{F} \,:\, A \cap \{\tau = n\} \in \mathcal{F}_n,\ n \in \mathbb{N}\}, \qquad \mathcal{F}_\tau = \{A \in \mathcal{F} \,:\, A \cap \{\tau \le n\} \in \mathcal{F}_n,\ n \in \mathbb{N}\}.$$

If $A \in \mathcal{F}_\tau$ and $n \in \mathbb{N}$ then $A \cap \{\tau = n\} = (A \cap \{\tau \le n\}) \cap (A \cap \{\tau \le n-1\})^c$. Now $A \cap \{\tau \le n\} \in \mathcal{F}_n$ and $A \cap \{\tau \le n-1\} \in \mathcal{F}_{n-1} \subseteq \mathcal{F}_n$ (because $A \in \mathcal{F}_\tau$). The sigma-algebra $\mathcal{F}_\tau$ is closed under taking complements and intersections, thus $A \cap \{\tau = n\} \in \mathcal{F}_n$. This holds for any $n \in \mathbb{N}$, thus $A \in \mathcal{F}'_\tau$.
Conversely, if $A \in \mathcal{F}'_\tau$, then $A \cap \{\tau \le n\} = \bigcup_{k=0}^{n}(A \cap \{\tau = k\})$, and $(A \cap \{\tau = k\}) \in \mathcal{F}_k \subseteq \mathcal{F}_n$ for all $k \le n$ (because $A \in \mathcal{F}'_\tau$), so the union of these events is also in $\mathcal{F}_n$, thus $A \cap \{\tau \le n\} \in \mathcal{F}_n$ for any $n \in \mathbb{N}$, thus $A \in \mathcal{F}_\tau$.

(c) For any Borel-set B we have $\{X_\tau \in B\} \in \mathcal{F}_\tau$, because for any $n \in \mathbb{N}$ we have

$$\{X_\tau \in B\} \cap \{\tau = n\} = \{X_n \in B\} \cap \{\tau = n\} \in \mathcal{F}_n,$$

since (X_n) is adapted to its own natural filtration and τ is stopping time.

(d) Let $(X_n)_{n \ge 0}$ be a discrete time Markov chain with natural filtration $(\mathcal{F}_n)$, τ be a stopping time, $s \in \mathbb{N}$ and $f : \mathbb{R} \to \mathbb{R}$ a bounded, Borel-measurable function. We want to check the strong Markov property, i.e., we want to show

$$\mathbb{E}(f(X_{\tau+s}) \,|\, \mathcal{F}_\tau) = \mathbb{E}(f(X_{\tau+s}) \,|\, \sigma(X_\tau)).$$

We are going to show that the right-hand-side satisfies the definition of the left-hand side. First we need to check that $\mathbb{E}(f(X_{\tau+s}) \,|\, \sigma(X_\tau))$ is $\mathcal{F}_\tau$-measurable, so we only need to check $\sigma(X_\tau) \subseteq \mathcal{F}_\tau$, which amounts to showing that X_τ is $\mathcal{F}_\tau$-measurable. Indeed, this was proved in (c).
In order to complete the proof of the fact that $\mathbb{E}(f(X_{\tau+s}) \,|\, X_\tau)$ satisfies the definition of the conditional expectation $\mathbb{E}(f(X_{\tau+s}) \,|\, \mathcal{F}_\tau)$, we only need to check that for any $A \in \mathcal{F}_\tau$ we have

$$\mathbb{E}(\mathbb{E}(f(X_{\tau+s}) \,|\, X_\tau)\mathbb{1}_A) = \mathbb{E}(f(X_{\tau+s})\mathbb{1}_A).$$

This is indeed the case:

$$\mathbb{E}(\mathbb{E}(f(X_{\tau+s}) \mid X_\tau)\mathbb{1}_A) = \sum_{n=0}^{\infty} \mathbb{E}(\mathbb{E}(f(X_{\tau+s}) \mid X_\tau)\mathbb{1}[A \cap \{\tau = n\}]) =$$

$$\sum_{n=0}^{\infty} \mathbb{E}(\mathbb{E}(f(X_{n+s}) \mid X_n)\mathbb{1}[A \cap \{\tau = n\}]) \overset{\text{Markov}}{=} \sum_{n=0}^{\infty} \mathbb{E}(\mathbb{E}(f(X_{n+s}) \mid \mathcal{F}_n)\mathbb{1}[A \cap \{\tau = n\}]) \overset{A \in \mathcal{F}_\tau}{=}$$

$$\sum_{n=0}^{\infty} \mathbb{E}(\mathbb{E}(f(X_{n+s})\mathbb{1}[A \cap \{\tau = n\}] \mid \mathcal{F}_n)) \overset{\text{tower}}{=} \sum_{n=0}^{\infty} \mathbb{E}(f(X_{n+s})\mathbb{1}[A \cap \{\tau = n\}]) = \mathbb{E}(f(X_{\tau+s})\mathbb{1}_A).$$

4. Let $(B(t))$ denote the standard Brownian motion. Show that for any $x \geq 0$ we have

$$\mathbb{P}\left(B(t) \geq x \;\middle|\; \min_{0\leq s\leq t} B(s) \geq 0\right) = \exp\left(\frac{-x^2}{2t}\right).$$

Hint: The probability of the condition is zero, so use the reflection principle (c.f. Lecture 10) to first show that

$$\mathbb{P}(B(t) \geq x - \varepsilon \mid \min_{0\leq s\leq t} B(s) \geq -\varepsilon) = \frac{\Phi\left(\frac{\varepsilon - x}{\sqrt{t}}\right) - \Phi\left(\frac{-\varepsilon - x}{\sqrt{t}}\right)}{2\Phi\left(\frac{\varepsilon}{\sqrt{t}}\right) - 1}, \qquad x > 0,\ \varepsilon > 0$$

and then let ε go to 0.

Solution. By the definition of conditional probability, we have

$$\mathbb{P}(B(t) \geq x - \varepsilon \mid \min_{0\leq s\leq t} B(s) \geq -\varepsilon) = \frac{\mathbb{P}(\{B(t) \geq x - \varepsilon\} \cap \{\min_{0\leq s\leq t} B(s) \geq -\varepsilon\})}{\mathbb{P}(\min_{0\leq s\leq t} B(s) \geq -\varepsilon)}.$$

Let us first deal with the denominator. Recall the claim from lecture 10:

$$\mathbb{P}(\min_{0\leq s\leq t} B(s) \geq -\varepsilon) = \mathbb{P}(\max_{0\leq s\leq t} B(s) \leq \varepsilon) = 2\Phi\left(\frac{\varepsilon}{\sqrt{t}}\right) - 1.$$

Now for the numerator,

$$\begin{aligned}
\mathbb{P}(\{B(t) \geq x - \varepsilon\} \cap \{\min_{0\leq s\leq t} B(s) \geq -\varepsilon\}) &= \mathbb{P}(\{B(t) \leq \varepsilon - x\} \cap \{\max_{0\leq s\leq t} B(s) \leq \varepsilon\}) = \\
&\mathbb{P}(B(t) \leq \varepsilon - x) - \mathbb{P}(\{B(t) \leq \varepsilon - x\} \cap \{\max_{0\leq s\leq t} B(s) \geq \varepsilon\}) \overset{(*)}{=} \\
&\Phi\left(\frac{\varepsilon - x}{\sqrt{t}}\right) - \mathbb{P}(\{B(t) \geq \varepsilon + x\} \cap \{\max_{0\leq s\leq t} B(s) \geq \varepsilon\}) = \\
&\Phi\left(\frac{\varepsilon - x}{\sqrt{t}}\right) - \mathbb{P}(B(t) \geq \varepsilon + x) = \Phi\left(\frac{\varepsilon - x}{\sqrt{t}}\right) - \Phi\left(\frac{-\varepsilon - x}{\sqrt{t}}\right),
\end{aligned}$$

where in the equation marked by $(*)$ we used the identity

$$\mathbb{P}(\{\max_{0\leq s\leq t} B(s) \geq \varepsilon\} \cap \{B(t) \leq \varepsilon - x\}) = \mathbb{P}(\{\max_{0\leq s\leq t} B(s) \geq \varepsilon\} \cap \{B(t) \geq \varepsilon + x\}),$$

which follows by reflecting the part of the trajectory of the Brownian motion that comes after hitting level ε.

Now let us take the limit $\varepsilon \to 0$. This is a $\frac{0}{0}$ type limit, so let's apply L'Hospital's rule.

$$\begin{aligned}
\lim_{\varepsilon\to 0} \frac{\Phi\left(\frac{\varepsilon - x}{\sqrt{t}}\right) - \Phi\left(\frac{-\varepsilon - x}{\sqrt{t}}\right)}{2\Phi\left(\frac{\varepsilon}{\sqrt{t}}\right) - 1} &= \lim_{\varepsilon\to 0} \frac{\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Phi\left(\frac{\varepsilon - x}{\sqrt{t}}\right) - \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Phi\left(\frac{-\varepsilon - x}{\sqrt{t}}\right)}{2\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Phi\left(\frac{\varepsilon}{\sqrt{t}}\right)} = \\
&\lim_{\varepsilon\to 0} \frac{\Phi'\left(\frac{\varepsilon - x}{\sqrt{t}}\right)\frac{1}{\sqrt{t}} + \Phi'\left(\frac{-\varepsilon - x}{\sqrt{t}}\right)\frac{1}{\sqrt{t}}}{2\Phi'\left(\frac{\varepsilon}{\sqrt{t}}\right)\frac{1}{\sqrt{t}}} = \frac{\Phi'\left(\frac{-x}{\sqrt{t}}\right) + \Phi'\left(\frac{-x}{\sqrt{t}}\right)}{2\Phi'\left(\frac{0}{\sqrt{t}}\right)} = \frac{2\frac{1}{\sqrt{2\pi t}}\exp(-\frac{(-x)^2}{2t})}{2\frac{1}{\sqrt{2\pi t}}\exp(-\frac{0^2}{2t})} = e^{\frac{-x^2}{2t}}
\end{aligned}$$

Thus

$$\mathbb{P}(B(t) \geq x \mid \min_{0\leq s\leq t} B(s) \geq 0) = \lim_{\varepsilon\to 0} \mathbb{P}(B(t) \geq x - \varepsilon \mid \min_{0\leq s\leq t} B(s) \geq -\varepsilon) = e^{\frac{-x^2}{2t}}.$$

Remark. The conditional law of Brownian motion on $[0, t]$ under the condition that it stays non-negative on $[0, t]$ is called the *Brownian meander*. What we have just found is that the value of the Brownian meander at time t has the *Rayleigh distribution*.