Stoch. Anal. HW assignment 5. Solutions

Note: Each of the 5 questions is worth 10 marks. Write the solutions of different exercises on different pages.

1. We approximate the stochastic integral $\mathcal{Z} = \int_0^1 (2B(t)+1)\mathrm{d}B(t)$ with the sum

$$\mathcal{Z}_n = \sum_{k=1}^{n} \left(2B\left(\frac{k-1}{n}\right)+1\right) \cdot \left(B\left(\frac{k}{n}\right) - B\left(\frac{k-1}{n}\right)\right).$$

   (a) Write down the simple predictable function $(X_n(t))$, $0 \le t \le 1$ for which $\mathcal{Z}_n = \int_0^1 X_n(t)\mathrm{d}B(t)$.

   (b) What is the smallest value of n for which $\mathbb{E}\left[(\mathcal{Z}-\mathcal{Z}_n)^2\right] \le 10^{-3}$ holds?

Solution:

(a) Let us define the partition $\Delta_n = \{t_0, \dots, t_n\}$ of the interval $[0,1]$ by setting $t_k = \frac{k}{n}$. Let $g(x) = 2x+1$. Then $\mathcal{Z}_n = \sum_{k=1}^n g\left(B\left(t_{k-1}\right)\right) \cdot \left(B\left(t_k\right) - B\left(t_{k-1}\right)\right)$ is a Stieltjes-type-sum where the sample point t_k^* is chosen as the left endpoint of the interval $[t_{k-1}, t_k]$. Thus, by the definition of stochastic integral for simple predictable processes (see Lecture 11), we have

$$\mathcal{Z}_n = \int_0^1 X_n(t)\mathrm{d}B(t), \qquad \text{where} \qquad X_n(t) = \sum_{k=1}^{n} g(B(t_{k-1}))\mathbb{1}[t_{k-1} < t \le t_k].$$

Also note that $(X_n(t))$ is indeed a simple predictable process, since $\xi_k = g(B(t_{k-1}))$ is indeed $\mathcal{F}_{t_{k-1}}$-measurable.

(b) $\mathcal{Z} - \mathcal{Z}_n$ is equal to $\int_0^1 (g(B_t) - X_n(t))\,\mathrm{d}B_t$ by the linearity of Itô integral.
$\mathbb{E}\left((\mathcal{Z}-\mathcal{Z}_n)^2\right) = \mathbb{E}\left[\int_0^1 (g(B_t)-X_n(t))^2\,\mathrm{d}t\right]$ by the Itô isometry.
$\mathbb{E}\left[\int_0^1 (g(B_t)-X_n(t))^2\,\mathrm{d}t\right] = \int_0^1 \mathbb{E}\left[(g(B_t)-X_n(t))^2\right]\,\mathrm{d}t$ by Fubini's theorem.

$$\int_0^1 \mathbb{E}\left[(g(B_t)-X_n(t))^2\right]\,\mathrm{d}t = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \mathbb{E}\left[(g(B_t)-X_n(t))^2\right]\,\mathrm{d}t =$$
$$\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \mathbb{E}\left[(g(B(t))-g(B(t_{k-1})))^2\right]\,\mathrm{d}t = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \mathbb{E}\left[(2B(t)-2B(t_{k-1}))^2\right]\,\mathrm{d}t =$$
$$4\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \mathbb{E}\left[(B(t)-B(t_{k-1}))^2\right]\,\mathrm{d}t = 4\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (t-t_{k-1})\,\mathrm{d}t = 4\sum_{k=1}^{n} \frac{1}{2}(t_k-t_{k-1})^2$$

Now by $t_k = \frac{k}{n}$ we obtain $t_k - t_{k-1} = \frac{1}{n}$, thus $\mathbb{E}\left((\mathcal{Z}-\mathcal{Z}_n)^2\right) = 4 \cdot n \cdot \frac{1}{2}\frac{1}{n^2} = \frac{2}{n}$.
Thus the smallest value of n for which we can guarantee that $\mathbb{E}\left((\mathcal{Z}-\mathcal{Z}_n)^2\right) \le 10^{-3}$ is $n = 2000$.

Remark:

$$\mathcal{Z} = \int_0^1 (2B(t)+1)\mathrm{d}B(t) = 2\int_0^1 B(t)\mathrm{d}B(t) + \int_0^1 \mathrm{d}B(t) = 2\cdot\frac{1}{2}\left(B^2(1)-1\right) + B(1) = B^2(1)+B(1)-1,$$

see Lecture 13.

2. *Discrete Itô formula.* We have learnt Itô's formula is class:

$$f(B_t) - f(B_0) = \int_0^t f'(B_s)\,\mathrm{d}B_s + \frac{1}{2}\int_0^t f''(B_s)\,\mathrm{d}s.$$

The goal of this exercise is to find the discrete analogue of this formula.

Let $S_n = \xi_1 + \cdots + \xi_n$, where $\xi_1, \xi_2, \ldots,$ are i.i.d. and $\mathbb{P}(\xi_k = 1) = \mathbb{P}(\xi_k = -1) = \frac{1}{2}$, $k \geq 1$. Let $(\mathcal{F}_n)_{n\geq 0}$ denote the natural filtration of the simple random walk (S_n), i.e., $\mathcal{F}_n = \sigma(S_1, \ldots, S_n) = \sigma(\xi_1, \ldots, \xi_n)$.

Given some $f : \mathbb{Z} \to \mathbb{R}$ let us define

$$f^*(x) := \frac{f(x+1) - f(x-1)}{2}, \qquad f^{**}(x) := f(x+1) - 2f(x) + f(x-1)$$

Note that $f^*(x)$ can be viewed as a discrete analogue of $f'(x)$, because the slope of the line that passes through the points $(x-1, f(x-1))$ and $(x+1, f(x+1))$ is $f^*(x)$.

Note that $f^{**}(x)$ can be viewed as a discrete analogue of $f''(x)$ because the second derivative of the parabola that passes through the points $(x-1, f(x-1))$, $(x, f(x))$ and $(x+1, f(x+1))$ is $f^{**}(x)$.

(a) Let us define $X_n = f(S_n)$. Find the discrete Doob-Meyer decomposition of the process (X_n), i.e., write $X_n = A_n + M_n$, where (A_n) is a predictable process and (M_n) is a martingale with zero expectation. Explicitly state the formula for A_n using the function $f^{**}(\cdot)$.

(b) Write M_n as the discrete stochastic integral $(H \cdot S)_n$ of a predictable process (H_n) with respect to the martingale (S_n). Explicitly state the formula for H_n using the function $f^*(\cdot)$.

(c) Put the results of (a) and (b) together to obtain a formula for $f(S_n) - f(S_0)$ that looks like the discrete version of the right-hand side of Itô's formula.

Solution:

(a) Recalling the construction of discrete Doob-Meyer-decomposition from Lecture 4:

$$f(S_{n+1}) - f(S_n) = \Delta f(S_n) = \Delta X_n = \Delta M_n + \Delta A_n$$

$$\Delta A_n = \mathbb{E}[\Delta X_n \,|\mathcal{F}_n] = \mathbb{E}[\Delta f(S_n)\,|\mathcal{F}_n] = \mathbb{E}[f(S_{n+1}) - f(S_n)\,|\mathcal{F}_n] = \frac{1}{2}\Big(f(S_n+1) + f(S_n-1)\Big) - f(S_n) = \frac{1}{2}f^{**}(S_n) \quad (1)$$

Thus

$$A_n = \mathbb{E}[f(S_1)] + \sum_{i=2}^{n} \Delta A_{i-1} = \frac{1}{2}\left(f(1) + f(-1)\right) + \sum_{i=2}^{n} \frac{1}{2} f^{**}(S_{i-1}).$$

$$\Delta M_n = \Delta X_n - \mathbb{E}[\Delta X_n \,|\,\mathcal{F}_n] = X_{n+1} - X_n - \mathbb{E}[X_{n+1} - X_n \,|\,\mathcal{F}_n] = X_{n+1} - \mathbb{E}[X_{n+1}\,|\,\mathcal{F}_n] = f(S_{n+1}) - \mathbb{E}[f(S_{n+1})\,|\,\mathcal{F}_n] = f(S_{n+1}) - \frac{1}{2}\Big(f(S_n+1) + f(S_n-1)\Big)$$

(b) We have $S_{n+1} = S_n + \Delta S_n = S_n + \xi_{n+1}$, where $\xi_{n+1} = \pm 1$, thus in both cases we have

$$\Delta M_n = \frac{1}{2}\Big(f(S_n+1) - f(S_n-1)\Big)\xi_{n+1} = f^*(S_n)\xi_{n+1} = f^*(S_n)\Delta S_n, \quad (2)$$

thus $H_n = f^*(S_{n-1})$ and $M_n = M_n - M_0 = \sum_{i=1}^n \Delta M_{i-1} = \sum_{i=1}^n f^*(S_{i-1})\Delta S_{i-1} = (H \cdot S)_n$.

(c) We obtain the discrete Itô formula:

$$f(S_n) - f(S_0) = \sum_{k=0}^{n-1} \Delta f(S_k) = \sum_{k=0}^{n-1} \Delta M_n + \sum_{k=0}^{n-1} \Delta A_n \overset{(1),(2)}{=} \sum_{k=0}^{n-1} f^*(S_k)\Delta S_k + \frac{1}{2}\sum_{k=0}^{n-1} f^{**}(S_k)$$

Remark. We have earlier proved two special cases of the discrete Itô formula: see the case $f(x) = x^2$ on Lecture 5 and the case $f(x) = e^{\lambda x}$ in Homework 2, exercise 1. Also note that a formula very similar to the discrete Itô formula appeared in the proof of the continuous Itô formula, see Lecture 15.

3. We say that two stochastic processes $(X(t))_{t\geq 0}$ and $(Y(t))_{t\geq 0}$ have the same law if for every choice of $n \geq 1$ and $0 \leq t_1 < t_2 < \cdots < t_n$ the joint distributions of $(X(t_1), X(t_2), \ldots, X(t_n))$ and $(Y(t_1), Y(t_2), \ldots, Y(t_n))$ are the same. Denote by $(B(t))$ the standard Brownian motion.

   Given $\beta \in (-\frac{1}{2}, +\infty)$, find $\alpha \in \mathbb{R}_+$ and $c > 0$ so that $(X(t))_{t\geq 0}$ and $(Y(t))_{t\geq 0}$ have the same law, where

$$X(t) = B(c \cdot t^\alpha), \qquad Y(t) = \int_0^t s^\beta \, \mathrm{d}B(s).$$

   Briefly explain why $(X(t))_{t\geq 0}$ and $(Y(t))_{t\geq 0}$ have the same law using results seen in class.

   Hint: Gaussian processes.

Solution:

(X_t) is a Gaussian process because Brownian motion is a Gaussian process and (X_t) arises from Brownian motion by a deterministic time-change. (Y_t) is a Gaussian process because it is an Itô integral where the integrand is a deterministic function, see lecture 13.

Thus, in order to find the parameters for which $(X(t))_{t\geq 0}$ and $(Y(t))_{t\geq 0}$ have the same law, we only need to guarantee that we have

$$\mathbb{E}(X_t) = \mathbb{E}(Y_t) \qquad \text{and} \qquad \mathrm{Cov}(X_s, X_t) = \mathrm{Cov}(Y_s, Y_t) \qquad \text{for any} \qquad 0 \leq s \leq t.$$

Now $\mathbb{E}(X_t) = 0$ because $B(c \cdot t^\alpha) \sim \mathcal{N}(0, c \cdot t^\alpha)$ and $\mathbb{E}(Y_t) = 0$ because an Itô integral has zero expectation, see lecture 12.

$\mathrm{Cov}(X_s, X_t) = c \cdot s^\alpha \wedge c \cdot t^\alpha = c \cdot s^\alpha$.

By the covariance formula learnt on lecture 13, we have

$$\mathrm{Cov}(Y_s, Y_t) = \int_0^s u^\beta \cdot u^\beta \, \mathrm{d}u = \left[\frac{u^{2\beta+1}}{2\beta+1} \right]_0^s = \frac{s^{2\beta+1}}{2\beta+1}$$

Thus if we choose $\alpha := 2\beta + 1$ (indeed $\alpha > 0$) and $c := \frac{1}{2\beta+1}$, then $\mathrm{Cov}(X_s, X_t) = \mathrm{Cov}(Y_s, Y_t)$ holds.

Remark. This exercise was a very simple special case of the *Dubins-Schwarz theorem* which (informally) states that every continuous martingale is the time-change of a standard Brownian motion. Here (Y_t) is the continuous martingale and (X_t) is the time-change of Brownian motion. However, in the case of the general Dubins-Schwarz theorem, the time-change is not necessarily deterministic, like in this exercise.

4. *Equivalent definitions of the Brownian bridge.* Denote by $(B(t))$ the standard Brownian motion. Briefly argue that the three processes below (each one is defined for $0 \le t \le 1$) are Gaussian and that the three processes have the same law.

   (a) $B(t) - tB(1)$.

   (b) $(1-t)B(t/(1-t))$.

   (c) $(1-t)\int_0^t \frac{1}{1-s}\,\mathrm{d}B_s$

Solution. We are going to show that the three processes are Gaussian and have the same mean function $\mu(\cdot)$ and autocovariance function $\gamma(\cdot,\cdot)$.

(a) This is a Gaussian process, because it arises from $(B(t))$ using linear operations.

$$\mu(t) = \mathbb{E}(B(t) - tB(1)) = \mathbb{E}(B(t)) - t\mathbb{E}(B(1)) = 0 - 0 = 0, \qquad 0 \le t \le 1.$$

Now if $0 \le s \le t \le 1$ then

$$\gamma(s,t) = \mathbf{Cov}(B(s) - sB(1),\, B(t) - tB(1)) = \\ \mathbf{Cov}(B(s), B(t)) - \mathbf{Cov}(B(s), tB(1)) - \mathbf{Cov}(sB(1), B(t)) + \mathbf{Cov}(sB(1), tB(1)) = \\ s - ts - st + st = s \cdot (1-t).$$

(b) This is again a Gaussian process, because it arises from $(B(t))$ using deterministic time-change and linear operations. We only have to check the expected values and the covariances.

$$\mu(t) = \mathbb{E}\left((1-t)B\left(\frac{t}{1-t}\right)\right) = (1-t)\cdot 0 = 0, \qquad 0 \le t \le 1.$$

Now if $0 \le s \le t \le 1$ then $\frac{s}{1-s} \le \frac{t}{1-t}$, and therefore

$$\gamma(s,t) \;=\; \mathbf{Cov}\left((1-s)B\left(\frac{s}{1-s}\right),\,(1-t)B\left(\frac{t}{1-t}\right)\right) \;=\; (1-s)(1-t)\frac{s}{1-s} \;=\; s\cdot(1-t).$$

(c) The process $(X_t)_{0\le t\le 1}$ defined by $X_t = \int_0^t \frac{1}{1-s}\,\mathrm{d}B_s$ is Gaussian (because it is an Itō integral of a deterministic function against Brownian motion, see lecture 13), thus $((1-t)X_t)_{0\le t\le 1}$ is also a Gaussian process. $\mu(t) = (1-t)\mathbb{E}(X_t) = 0$ for $0 \le t \le 1$ and for $0 \le s \le t \le 1$ we have

$$\gamma(s,t) = \mathbf{Cov}\left((1-s)\int_0^s \frac{1}{1-u}\,\mathrm{d}B_u,\,(1-t)\int_0^t \frac{1}{1-u}\,\mathrm{d}B_u\right) = \\ (1-s)(1-t)\int_0^s \frac{1}{(1-u)^2}\,\mathrm{d}u = (1-s)(1-t)\frac{s}{1-s} = s\cdot(1-t).$$

Remark. We will learn more about the Brownian bridge in class. For example, the conditional law of $(B_t)_{0\le t\le 1}$ under the condition that $B_1 = 0$ is the same as the (unconditioned) law of $(B_t - tB_1)_{0\le t\le 1}$, see lecture 17.

5. Give an explicit formula for the cumulative distribution function $F(x) = \mathbb{P}(X \leq x),\ x \in \mathbb{R}$ of the random variable X, where

$$X = \int_0^1 \frac{e^{B_u}\,\mathrm{d}u}{2(e^{B_u}+1)^2} + \int_0^1 \frac{\mathrm{d}B_u}{1+e^{-B_u}}.$$

Hint: You may use $\Phi(\cdot)$, the c.d.f. of the standard normal distribution in your solution.

Solution.

We observe that if $f(x) = \log(e^x + 1)$ then $f'(x) = \frac{1}{1+e^{-x}}$ and $f''(x) = \frac{e^{-x}}{(1+e^{-x})^2} = \frac{e^x}{(1+e^x)^2}$.

So by Itō's formula,

$$X = \int_0^1 f'(B_u)\mathrm{d}B_u + \frac{1}{2}\int_0^1 f''(B_u)\mathrm{d}u = f(B_1) - f(B_0) = \log(e^{B_1}+1) - \log 2.$$

Now the cumulative distribution function (c.d.f.) of X is

$$F(x) = \mathbb{P}(X \leq x) = \mathbb{P}(\log(e^{B_1}+1) - \log 2 \leq x) = \mathbb{P}(B_1 \leq \log(e^{x+\log 2} - 1)) = \Phi(\log(e^{x+\log 2} - 1)).$$