## Stoch. Anal. HW assignment 6., Solutions

1. *Multidimensional Brownian motion.* We say that $\big(\underline{B}(t)\big)$ is a $d$-dimensional Brownian motion if

$$\underline{B}(t) = \big(B_1(t), \dots, B_d(t)\big)^T,$$

where $(B_1(t)), \dots, (B_d(t))$ are independent one-dimensional standard Brownian motions. Thus $\big(\underline{B}(t)\big)$ is an $\mathbb{R}^d$-valued stochastic process, i.e., a random column vector that evolves in time.

Also, $\underline{B} : \mathbb{R}_+ \to \mathbb{R}^d$ can be viewed as a random vector-valued continuous function.

We say that the $d$-dimensional matrix $\underline{\underline{U}} = (U_{i,j})_{i,j=1}^d$ is orthogonal if $\underline{\underline{U}}^T \underline{\underline{U}} = \underline{\underline{U}}\,\underline{\underline{U}}^T = I$, where $I$ is the $d$-dimensional identity matrix. That is, the rows of $\underline{\underline{U}}$ form an orthonormal basis of $\mathbb{R}^d$. Also, the columns of $\underline{\underline{U}}$ form an orthonormal basis of $\mathbb{R}^d$. The linear transformation $\underline{x} \mapsto \underline{\underline{U}}\,\underline{x}$ is an isometry, i.e., the length of $\underline{\underline{U}}\,\underline{x}$ is equal to the length of $\underline{x}$. Typical examples of such transformations include reflections and rotations.

Show that if $\underline{\underline{U}}$ is a $d$-dimensional *orthogonal matrix* and if we define

$$\widetilde{\underline{B}}(t) = \underline{\underline{U}}\,\underline{B}(t), \quad t \in \mathbb{R}_+,$$

then $\Big(\widetilde{\underline{B}}(t)\Big)$ is also a $d$-dimensional Brownian motion.

*Hint:* You only need to check that $(\widetilde{B}_1(t)), \dots, (\widetilde{B}_d(t))$ are independent one-dimensional standard Brownian motions. You might want to use the equivalent definition of Brownian motion as a Gaussian process.

**Solution:**

We have $\widetilde{B}_i(t) = \sum_{j=1}^d U_{i,j} B_j(t)$, where $\sum_{j=1}^d U_{i,j}^2 = 1$ and $\sum_{j=1}^d U_{i,j} U_{\ell,j} = 0$ if $i \neq \ell$.

In order to check that $\Big(\widetilde{\underline{B}}(t)\Big)$ is a $d$-dimensional Brownian motion, we need to check that $\widetilde{B}_1(t), \dots, \widetilde{B}_d(t)$ are also independent standard Brownian motions.

First we will check that $(B_i(t))$ is indeed a Brownian motion. Clearly $(B_i(t))$ is continuous and

$$\mathbb{E}[\widetilde{B}_i(t)] = \sum_{j=1}^d U_{i,j} \mathbb{E}[B_j(t)] = \sum_{j=1}^d U_{i,j} \cdot 0 = 0,$$

$$\mathrm{Cov}(\widetilde{B}_i(s), \widetilde{B}_i(t)) = \mathrm{Cov}\left(\sum_{j=1}^d U_{i,j} B_j(s), \sum_{k=1}^d U_{i,k} B_k(t)\right) \overset{(*)}{=}$$

$$\sum_{j=1}^d \mathrm{Cov}(U_{i,j} B_j(s), U_{i,j} B_j(t)) = \sum_{j=1}^d U_{i,j}^2 \cdot (s \wedge t) = s \wedge t,$$

where in $(*)$ we used bilinearity and the fact that $(B_1(t)), \dots, (B_d(t))$ are independent. Thus, by the equivalent characterization of Brownian motion as a Gaussian process (see lecture 7) we see that $(\widetilde{B}_i(t))$ is a Brownian motion.

Now we will check that $\widetilde{B}_1(t), \dots, \widetilde{B}_d(t)$ are independent. Since we are dealing with Gaussian processes, it is enough to check that if $i \neq \ell$, then $\mathrm{Cov}(\widetilde{B}_i(s), \widetilde{B}_\ell(t)) = 0$. Indeed:

$$\mathrm{Cov}(\widetilde{B}_i(s), \widetilde{B}_\ell(t)) = \mathrm{Cov}\left(\sum_{j=1}^d U_{i,j} B_j(s), \sum_{k=1}^d U_{\ell,k} B_k(t)\right) \overset{(*)}{=}$$

$$\sum_{j=1}^d \mathrm{Cov}(U_{i,j} B_j(s), U_{\ell,j} B_j(t)) = \sum_{j=1}^d U_{i,j} U_{\ell,j} \cdot (s \wedge t) = 0,$$

where in $(*)$ we used bilinearity and the fact that $(B_1(t)), \dots, (B_d(t))$ are independent.

2. (a) Use Itō calculus to show that

$$M_2(t) = B_t^2 - t, \qquad M_4(t) = B_t^4 - 6tB_t^2 + 3t^2$$

are martingales. *Hint:* First calculate the stochastic differential of $(M_2(t))$ and $(M_4(t))$.

(b) Given $a \in \mathbb{R}_+$ let us define the stopping time

$$\tau = \min\{\, t \,:\, |B_t| = a \,\}.$$

Use the optional stopping theorem to calculate $\mathbb{E}(\tau)$ and $\mathbb{E}(\tau^2)$.
*Instruction:* You don't have to check that the optional stopping thm can be applied here – we will do that in class.

**Solution.**

(a) Using Itô's formula with $f(x) = x^2$, we obtain $\mathrm{d}M_2(t) = 2B_t\mathrm{d}B_t + \frac{1}{2}2\mathrm{d}t - \mathrm{d}t = 2B_t\mathrm{d}B_t$, thus $M_2(t) = M_2(t) - M_2(0) = \int_0^t 2B_s\mathrm{d}B_s$ which is an Itō-integral w.r.t. Brownian motion, hence a martingale.
In the calculation below we use Itô's formula with $f(x) = x^4$ as well as the product rule of stochastic differentials (in the case when one term of the product is a function with finite total variation, thus the mutual variation term vanishes):

$$\begin{aligned}
\mathrm{d}M_4(t) &= \mathrm{d}B_t^4 - 6\mathrm{d}(tB_t^2) + 3\mathrm{d}(t^2),\\
&= 4B_t^3\mathrm{d}B_t + \frac{1}{2}\cdot 12B_t^2\mathrm{d}t - 6(\mathrm{d}tB_t^2 + t\mathrm{d}(B_t^2) + 0) + 3\cdot 2t\mathrm{d}t,\\
&= 4B_t^3\mathrm{d}B_t + 6B_t^2\mathrm{d}t - 6\mathrm{d}tB_t^2 - 6t(2B_t\mathrm{d}B_t + \frac{1}{2}\cdot 2\mathrm{d}t) + 6t\mathrm{d}t,\\
&= 4B_t^3\mathrm{d}B_t - 12tB_t\mathrm{d}B_t,\\
&= (4B_t^3 - 12tB_t)\mathrm{d}B_t + 0\mathrm{d}t.
\end{aligned}$$

Writing this in integral form:

$$M_4(t) = M_4(t) - M_4(0) = 4\int_0^t (B_s^3 - 3sB_s)\mathrm{d}B_s,$$

which is an Itō-integral w.r.t. Brownian motion, hence a martingale.

(b) $0 = M_2(0) = \mathbb{E}[M_2(\tau)] = \mathbb{E}[B_\tau^2 - \tau] = \mathbb{E}[B_\tau^2] - \mathbb{E}[\tau] = a^2 - \mathbb{E}[\tau]$, thus $\mathbb{E}[\tau] = a^2$.

$$\begin{aligned}
0 = M_4(0) = \mathbb{E}[M_4(\tau)] = \mathbb{E}[B_\tau^4 - 6\tau B_\tau^2 + 3\tau^2] =\\
\mathbb{E}[a^4 - 6\tau a^2 + 3\tau^2] = a^4 - 6\mathbb{E}[\tau]a^2 + 3\mathbb{E}[\tau^2] = -5a^4 + 3\mathbb{E}[\tau^2].
\end{aligned}$$

Thus $\mathbb{E}[\tau^2] = \frac{5}{3}a^4$.

**Remark.** In order to see why we can apply the optional stopping theorem here, see lecture 21.
We also remark that it is not surprising that $\mathbb{E}[\tau_a^2] = c\cdot a^4$ for some constant $c$, where

$$\tau_a = \min\{\, t \,:\, |B_t| = a \,\},$$

as we now explain. The scale invariance of Brownian motion (see lecture 7) implies that $\tau_a$ has the same distribution as $a^2\cdot\tau_1$, thus $\mathbb{E}[\tau_a^2] = \mathbb{E}[(a^2\cdot\tau_1)^2] = a^4\mathbb{E}[\tau_1^2]$.

3. Find a simple explicit formula for a non-negative process $(X_t)$ satisfying

$$\mathrm{d}X_t = 4X_t\mathrm{d}B_t + 2X_t\mathrm{d}t, \qquad X_0 = 5.$$

*Hint:* First calculate the stochastic differential of $Y_t := \log(X_t)$ using Itô's formula for Itô processes and find a simple explicit formula expressing $Y_t$ in terms of $B_t$ and $t$.

**Solution.**

If $f(x) = \log(x)$ then $f'(x) = \frac{1}{x}$ and $f''(x) = -\frac{1}{x^2}$.

If $\mathrm{d}X_t = 4X_t\mathrm{d}B_t + 2X_t\mathrm{d}t$ then $\mathrm{d}[X]_t = \sigma_t^2\mathrm{d}t = (4X_t)^2\mathrm{d}t = 16X_t^2\mathrm{d}t$.

Thus

$$\mathrm{d}\log(X_t) = \mathrm{d}f(X_t) = f'(X_t)\mathrm{d}X_t + \frac{1}{2}f''(X_t)\mathrm{d}[X]_t =$$
$$\frac{1}{X_t}\left(4X_t\mathrm{d}B_t + 2X_t\mathrm{d}t\right) + \frac{1}{2}\left(-\frac{1}{X_t^2}\right)16X_t^2\mathrm{d}t = 4\mathrm{d}B_t + 2\mathrm{d}t - 8\mathrm{d}t = 4\mathrm{d}B_t - 6\mathrm{d}t,$$

therefore

$$\log(X_t) - \log(X_0) = \int_0^t \mathrm{d}\log(X_s) = \int_0^t 4\mathrm{d}B_s - \int_0^t 6\mathrm{d}s = 4B_t - 4B_0 - 6t = 4B_t - 6t$$

Exponentiating both sides we obtain

$$\frac{X_t}{X_0} = e^{4B_t - 6t}$$

Now $X_0 = 5$, thus

$$X_t = 5e^{4B_t - 6t}.$$

**Remark.** We have just solved a linear stochastic differential equation (SDE)! In particular, this exercise was about a special case of *geometric Brownian motion*, which is itself a special case of *stochastic exponentials*, which is itself a special case of general linear SDEs.

We will solve more SDE's in class.

4. Recall the definition of the Ornstein-Uhlenbeck process from lecture 13:

$$Y_t = e^{-\beta t} Y_0 + \sqrt{2\beta} \int_0^t e^{\beta \cdot (u-t)} \, \mathrm{d}B_u$$

(a) Calculate the stochastic differential of $Y_t$ and show that $(Y_t)$ is an Itô process by explicitly writing down the formula of the processes $(\mu_t)$ and $(\sigma_t)$ for which

$$Y_t = Y_0 + \int_0^t \mu_s \, \mathrm{d}s + \int_0^t \sigma_s \, \mathrm{d}B_s.$$

(b) Calculate the quadratic variation $[Y]_t$ of the O.-U. process on the interval $[0, t]$.

**Solution.**

(a) $Y_t = Y_0 e^{-\beta t} + \sqrt{2\beta} e^{-\beta t} \int_0^t e^{\beta u} \, \mathrm{d}B_u$, thus

$$\begin{aligned}
\mathrm{d}Y_t = Y_0 \cdot \mathrm{d}e^{-\beta t} + \sqrt{2\beta} \cdot \mathrm{d}\left( e^{-\beta t} \int_0^t e^{\beta u} \, \mathrm{d}B_u \right) \overset{(*)}{=} \\
Y_0 \left( -\beta e^{-\beta t} \right) \mathrm{d}t + \sqrt{2\beta} \cdot \left( (\mathrm{d}e^{-\beta t}) \int_0^t e^{\beta u} \, \mathrm{d}B_u + e^{-\beta t} \cdot \mathrm{d}\left( \int_0^t e^{\beta u} \, \mathrm{d}B_u \right) \right) = \\
- \beta Y_0 e^{-\beta t} \mathrm{d}t + \sqrt{2\beta} \cdot \left( (-\beta e^{-\beta t} \mathrm{d}t) \int_0^t e^{\beta u} \, \mathrm{d}B_u + e^{-\beta t} \cdot e^{\beta t} \mathrm{d}B_t \right) = \\
\left( -\beta Y_0 e^{-\beta t} - \beta \sqrt{2\beta} e^{-\beta t} \int_0^t e^{\beta u} \, \mathrm{d}B_u \right) \mathrm{d}t + \sqrt{2\beta} \mathrm{d}B_t = -\beta Y_t \mathrm{d}t + \sqrt{2\beta} \mathrm{d}B_t
\end{aligned}$$

where in the equation marked by $(*)$ we used the stochastic product rule and the fact that the mutual variation of a smooth function and a continuous function is zero. Thus

$$Y_t = Y_0 + \int_0^t \mu_s \, \mathrm{d}s + \int_0^t \sigma_s \, \mathrm{d}B_s,$$

where

$$\mu_t = -\beta Y_t, \qquad \sigma_t = \sqrt{2\beta}.$$

(b) $[Y]_t = \int_0^t \sigma_s^2 \mathrm{d}s = \int_0^t 2\beta \mathrm{d}s = 2\beta t$.

**Remark.** We observe that $\mathrm{d}Y_t = -\beta Y_t \mathrm{d}t + \sqrt{2\beta} \mathrm{d}B_t$, in other words the O.-U. process solves a certain linear stochastic differential equation. We will discuss more examples of SDE's in class.

5. *Discrete Tanaka's formula.*

(a) Let $x_0 \in \mathbb{Z}$. Write down the special case of the discrete Itô formula (see the solution of HW 5.2) when the function $f$ is chosen as $f(x) = |x - x_0|$. *Hint:* What is $f^*(x)$ and $f^{**}(x)$ in this case?

(b) Denote by $(S_n)$ the one dimensional simple symmetric random walk starting from $S_0 = 0$. Use (a) and the optional stopping theorem to calculate the expected number of steps that the walker spends at location $x_0 = 5$ before it reaches either level $-5$ or level 20.

**Solution.**

(a) Now $f(x) = |x - x_0|$. We first calculate

$$f^*(x) := \frac{f(x+1) - f(x-1)}{2} = \operatorname{sgn}(x - x_0) = \begin{cases} 1 & \text{if} \quad x > x_0, \\ 0 & \text{if} \quad x = x_0, \\ -1 & \text{if} \quad x < x_0. \end{cases}$$

$$f^{**}(x) := f(x+1) - 2f(x) + f(x-1) = 2\mathbb{1}[x = x_0] = \begin{cases} 2 & \text{if} \quad x = x_0, \\ 0 & \text{if} \quad x \neq x_0, \end{cases}$$

Thus, assuming $S_0 = 0$ the discrete Itô formula becomes

$$|S_n - x_0| - |x_0| = \sum_{k=0}^{n-1} \operatorname{sgn}(S_k - x_0)\Delta S_k + \sum_{k=0}^{n-1} \mathbb{1}[S_k = x_0].$$

(b) Let us define $\tau = T_{-5} \wedge T_{20}$, thus $\tau$ is the stopping time when the walker reaches either level $-5$ or level 20. Thus we want to calculate $\mathbb{E}\left(\sum_{k=0}^{\tau-1} \mathbb{1}[S_k = 5]\right)$. Now we note that

$$M_n = |S_n - 5| - 5 - \sum_{k=0}^{n-1} \mathbb{1}[S_k = 5] = \sum_{k=0}^{n-1} \operatorname{sgn}(S_k - 5)\Delta S_k$$

is a martingale, because it is a discrete stochastic integral of a predictable process w.r.t. $(S_n)$, see lecture 2.

We want to apply the optional stopping theorem, i.e., we want to conclude that $\mathbb{E}(M_\tau) = \mathbb{E}(M_0)$, but the stopped process $(M_{\tau \wedge n})$ is not bounded, so we cannot directly apply the theorem learnt on lecture 4. But it is not hard to check directly that $\mathbb{E}(M_\tau) = \mathbb{E}(M_0)$, as we now argue.

The stopped process $(M_{\tau \wedge n})$ is also a martingale (see lecture 4), thus $\mathbb{E}[M_{\tau \wedge n}] = M_0 = 0$.

This implies $\mathbb{E}\left(|S_{\tau \wedge n} - 5|\right) - 5 = \mathbb{E}\left(\sum_{k=0}^{\tau \wedge n - 1} \mathbb{1}[S_k = 5]\right)$. We know that $\mathbb{P}(\tau < \infty) = 1$.

Now $|S_{\tau \wedge n} - 5| \leq 15$, so by the dom. conv. thm we have $\lim_{n\to\infty} \mathbb{E}\left(|S_{\tau \wedge n} - 5|\right) = \mathbb{E}\left(|S_\tau - 5|\right)$. By the mon. conv. thm, we have $\lim_{n\to\infty} \mathbb{E}\left(\sum_{k=0}^{\tau \wedge n - 1} \mathbb{1}[S_k = 5]\right) = \mathbb{E}\left(\sum_{k=0}^{\tau-1} \mathbb{1}[S_k = 5]\right)$.

We have verified that $\mathbb{E}(M_\tau) = \mathbb{E}(M_0)$, or, equivalently,

$$\mathbb{E}\left(\sum_{k=0}^{\tau-1} \mathbb{1}[S_k = 5]\right) = \mathbb{E}\left(|S_\tau - 5|\right) - 5.$$

Now, similarly to the solution of the 2016 midterm 1, question 1(a), we have

$$\mathbb{P}(S_\tau = T_{-5}) = \frac{20}{5 + 20} = \frac{4}{5}, \qquad \mathbb{P}(S_\tau = T_{20}) = \frac{5}{5 + 20} = \frac{1}{5},$$

thus

$$\mathbb{E}\left(\sum_{k=0}^{\tau-1} \mathbb{1}[S_k = 5]\right) = \left(\frac{4}{5}\left|-5 - 5\right| + \frac{1}{5}\left|20 - 5\right|\right) - 5 = 6.$$

**Remark.** Tanaka's formula states that for any $x_0 \in \mathbb{R}$,

$$|B_t - x_0| - |B_0 - x_0| = \int_0^t \operatorname{sgn}(B_s - x_0)\, \mathrm{d}B_s + L(t, x_0),$$

where $L(t, x_0)$ is the so-called *local time* spent by the Brownian motion at location $x_0$ before time $t$. We will learn the definition of the local time and the proof of Tanaka's formula soon.