Stoch. Anal. HW assignment 7. Solution

1. *Hermite polynomials.* We have already shown that the processes defined by

$$M_1(t) = B_t, \qquad M_2(t) = B_t^2 - t, \qquad M_3(t) = B_t^3 - 3tB_t, \qquad M_4(t) = B_t^4 - 6tB_t^2 + 3t^2$$

are martingales.

(a) Let us define $p_n(t,x) := \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \exp(\lambda x - \frac{\lambda^2}{2}t)\Big|_{\lambda=0}$. Calculate $p_n(t,x)$ for $n = 1,2,3,4$.

(b) We say that the function $u(t,x)$ is a solution of the reverse heat equation if $\frac{\partial}{\partial t}u(t,x) \equiv -\frac{1}{2}\frac{\partial^2}{\partial x^2}u(t,x)$. Show that for any value of the parameter $\lambda \in \mathbb{R}$, the function $\exp(\lambda x - \frac{\lambda^2}{2}t)$ solves the reverse heat equation and that the polynomials $p_n(t,x), n \in \mathbb{N}$ inherit this property.

(c) Show that for any $n \geq 0$ the process defined by $M_n(t) := p_n(t, B_t)$ is a martingale.

Solution.

(a)

$$p_1(t,x) = x, \quad p_2(t,x) = x^2 - t, \quad p_3(t,x) = x^3 - 3tx, \quad p_4(t,x) = x^4 - 6x^2t + 3t^2.$$

(b)

$$\frac{\partial}{\partial t}\exp\left(\lambda x - \frac{\lambda^2}{2}t\right) = -\frac{\lambda^2}{2}\exp\left(\lambda x - \frac{\lambda^2}{2}t\right) = -\frac{1}{2}\frac{\partial^2}{\partial x^2}\exp\left(\lambda x - \frac{\lambda^2}{2}t\right)$$

The polynomials $p_n(t,x)$ indeed inherit this property, because

$$\frac{\partial}{\partial t}p_n(t,x) = \frac{\partial}{\partial t}\left(\frac{\mathrm{d}}{\mathrm{d}\lambda^n}\exp\left(\lambda x - \frac{\lambda^2}{2}t\right)\big|_{\lambda=0}\right) = \frac{\mathrm{d}}{\mathrm{d}\lambda^n}\left(\frac{\partial}{\partial t}\exp\left(\lambda x - \frac{\lambda^2}{2}t\right)\right)\big|_{\lambda=0}$$
$$= \frac{\mathrm{d}}{\mathrm{d}\lambda^n}\left(-\frac{1}{2}\frac{\partial^2}{\partial x^2}\exp\left(\lambda x - \frac{\lambda^2}{2}t\right)\right)\big|_{\lambda=0} = -\frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\frac{\mathrm{d}}{\mathrm{d}\lambda^n}\exp\left(\lambda x - \frac{\lambda^2}{2}t\right)\big|_{\lambda=0}\right) = -\frac{1}{2}\frac{\partial^2}{\partial x^2}p_n(t,x).$$

(c) By part (c) we know that

$$p_n^{(t)}(t,B_t) + \frac{1}{2}p_n^{(xx)}(t,B_t) = 0.$$

Now let us use the time-dependent Itō-formula for $p_n(t,B_t)$.

$$\mathrm{d}p_n(t,B_t) = p_n^{(t)}(t,B_t)\mathrm{d}t + p_n^{(x)}(t,B_t)\mathrm{d}B_t + \frac{1}{2}p_n^{(xx)}(t,B_t)\underbrace{\mathrm{d}[B]_t}_{\mathrm{d}t}$$
$$= p_n^{(x)}(t,B_t)\mathrm{d}B_t.$$

Writing it in integral form and noting that $p_n(0,0) = 0$ we obtain

$$M_n(t) = p_n(t,B_t) = p_n(t,B_t) - p_n(0,B_0) = \int_0^t p_n^{(x)}(s,B_s)\mathrm{d}B_s,$$

and this is an Itō-integral w.r.t. Brownian motion, hence a martingale.

2. (a) Show that $\widehat{M}_t = t^2 B_t - 2\int_0^t sB_s\,\mathrm{d}s$ is a martingale by writing it as an Itō integral w.r.t. (B_t).

   (b) Show that if $p(t,x)$ is a polynomial and

$$r(t,x) = \left(\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right) p(t,x)$$

   then $\widetilde{M}_t$ is a martingale, where

$$\widetilde{M}_t = p(t,B_t) - \int_0^t r(s,B_s)\,\mathrm{d}s.$$

   (c) Find the Doob-Meyer decomposition $X_t = A_t + M_t$ of $X_t = (B_t^2 - t)^2$, where A_t is an adapted process with finite total variation and M_t is a martingale. Write M_t as an Itō integral w.r.t. Brownian motion.

Solution.

(a) By either the time-dependent Itō-formula or by integration by parts we have

$$t^2 B_t - 0 = \int_0^t 2sB_s\mathrm{d}s + \int_0^t s^2\mathrm{d}B_s + 0.$$

Hence

$$t^2 B_t - \int_0^t 2sB_s\mathrm{d}s = \int_0^t s^2\mathrm{d}B_s.$$

(b) Let us use the time-dependent Itō-formula for $p(t,B_t)$.

$$p(t,B_t) - p(0,B_0) = \int_0^t \frac{\partial}{\partial t}p(s,B_s)\mathrm{d}s + \int_0^t \frac{\partial}{\partial x}p(s,B_s)\mathrm{d}B_s + \frac{1}{2}\int_0^t \frac{\partial^2}{\partial x^2}p(s,B_s)\underbrace{\mathrm{d}[B]_s}_{\mathrm{d}s},$$

$$p(t,B_t) - \int_0^t \left(\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right) p(s,B_s)\mathrm{d}s - p(0,B_0) = \int_0^t \frac{\partial}{\partial x}p(s,B_s)\mathrm{d}B_s,$$

$$\widetilde{M}_t - \widetilde{M}_0 = \int_0^t \frac{\partial}{\partial x}p(s,B_s)\mathrm{d}B_s,$$

hence $\widetilde{M}_t$ is a martingale.

(c) If $p(t,x) = (x^2 - t)^2$ then $X_t = p(t,B_t)$ and we may use the notation of the previous sub-exercise and calculate $\frac{\partial}{\partial x}p(t,x) = 2(x^2-t)\cdot 2x = 4(x^3 - xt)$

$$r(t,x) = -2(x^2-t) + \frac{1}{2}\frac{\partial}{\partial x}(4(x^3 - xt)) = -2(x^2-t) + 6x^2 - 2t = 4x^2,$$

thus the Doob-Meyer decomposition of X_t is

$$X_t = \widetilde{M}_t + \int_0^t 4B_s^2\mathrm{d}s.$$

We can write $\widetilde{M}_t$ as an Itō-integral w.r.t. Brownian motion in the following way:

$$\widetilde{M}_t = \int_0^t \frac{\partial}{\partial x}p(s,B_s)\mathrm{d}B_s = \int_0^t 4(B_s^3 - sB_s)\mathrm{d}B_s.$$

3. Given two independent Brownian motions $(B_1(t))$ and $(B_2(t))$ construct a third Brownian motion $(B_3(t))$ such that the mutual variation of $B_1(\cdot)$ and $B_3(\cdot)$ satisfies $[B_1, B_3]_t \equiv f(t)$, where $f : \mathbb{R}_+ \to \mathbb{R}$ is a given continuously differentiable function. What other conditions do we need about f for the exercise to be solvable?

Hint: Construct $(B_3(t))$ as

$$B_3(t) = \int_0^t g_1(u)\mathrm{d}B_1(u) + \int_0^t g_2(u)\mathrm{d}B_2(u) = Y_1(t) + Y_2(t)$$

for some appropriately chosen $g_1, g_2 : \mathbb{R}_+ \to \mathbb{R}$. Use what we have learnt about the quadratic/mutual variation of Itô processes driven by d-dimensional Brownian motion (lecture 21). Show that $(B_3(t))$ is a Brownian motion using one of the equivalent characterizations of B.M. that we have learnt in class.

Solution. We certainly need $f(0) = 0$ because $[B_1, B_3]_0 = 0$.

The increment of $B_3(t)$ is

$$\mathrm{d}B_3(t) = g_1(t)\mathrm{d}B_1(t) + g_2(t)\mathrm{d}B_2(t),$$

thus $B_3(t)$ is a martingale (the drift is zero). The increment of the quadratic variation of $B_3(t)$ is

$$\mathrm{d}[B_3]_t = (\mathrm{d}B_3(t))^2 = g_1^2(t)(\mathrm{d}B_1(t))^2 + 2g_1(t)g_2(t)\mathrm{d}B_1(t)\mathrm{d}B_2(t) + g_2^2(t)(\mathrm{d}B_2(t))^2 = \\ g_1^2(t)\mathrm{d}t + 2g_1(t)g_2(t)\cdot 0 + g_2^2(t)\mathrm{d}t = (g_1^2(t) + g_2^2(t))\mathrm{d}t.$$

According to Paul Levy's characterization (see lecture 22), we need $\mathrm{d}[B_3]_t = \mathrm{d}t$, thus if $g_1^2(t) + g_2^2(t) \equiv 1$ then $(B_3(t))$ is also a standard Brownian motion because $[B_3]_t = \int_0^t \mathrm{d}[B_3]_s = \int_0^t \mathrm{d}s = t$.

Now let us calculate the increment of the mutual variation of $(B_1(t))$ and $(B_3(t))$:

$$\mathrm{d}[B_1, B_3]_t = \mathrm{d}B_1(t)\mathrm{d}B_3(t) = \mathrm{d}B_1(t)(g_1(t)\mathrm{d}B_1(t) + g_2(t)\mathrm{d}B_2(t)) = \\ g_1(t)(\mathrm{d}B_1(t))^2 + g_2(t)\mathrm{d}B_1(t)\mathrm{d}B_2(t) = g_1(t)\mathrm{d}t + g_2(t)\cdot 0 = g_1(t)\mathrm{d}t.$$

We want $[B_1, B_3]_t = f(t)$, i.e.,

$$\mathrm{d}[B_1, B_3]_t = \mathrm{d}f(t) = f'(t)\mathrm{d}t$$

Thus is we choose

$$g_1(t) = f'(t), \qquad g_2(t) = \pm\sqrt{1 - (f'(t))^2}$$

then $g_1^2(t) + g_2^2(t) \equiv 1$ and

$$[B_1, B_3]_t = \int_0^t \mathrm{d}[B_1, B_3]_s = \int_0^t f'(s)\mathrm{d}s = f(t) - f(0) = f(t).$$

Note that we want the expression under the square root to be non-negative, thus the extra conditions that we need about f for the exercise to be solvable are:

$$f(0) = 0, \qquad f'(t) \in [-1, 1].$$

4. *Log-optimal portfolio.* You are trading shares at the stock market. The value of one stock at time t is

$$S_t = \sigma B_t + \mu t,$$

where (B_t) is standard Brownian motion and $\mu > 0$ (this is a toy example so let's not worry about the fact that S_t can become negative). At time t you hold C_t shares. Denote by $Y_0 > 0$ your initial wealth and by Y_t your total wealth at time t. It is OK to go in debt (i.e., it is OK if $C_t > Y_t$). Let us assume that you can't predict the future, so that (C_t) is left-continuous and adapted to $(\mathcal{F}_t)$, where $\mathcal{F}_t = \sigma(S_u,\, 0 \le u \le t)$. We have learnt in class (lecture 17) your net gain from trading shares at time t is

$$Y_t - Y_0 = \int_0^t C_u \,\mathrm{d}S_u.$$

You trade in stocks until time T. Your goal is to maximize your expected rate of return $\mathbb{E}\left[\log\left(\frac{Y_T}{Y_0}\right)\right]$. What is the maximal expected rate of return achievable and the trading strategy that achieves it?

Solution. We know that $\mathrm{d}Y_t = C_t \mathrm{d}S_t$ and $\mathrm{d}S_t = \sigma \mathrm{d}B_t + \mu \mathrm{d}t$. Therefore

$$\mathrm{d}\log(Y_t) = \log'(Y_t)\mathrm{d}Y_t + \frac{1}{2}\log''(Y_t)\mathrm{d}[Y]_t = \frac{C_t \mathrm{d}S_t}{Y_t} - \frac{1}{2}\frac{C_t^2 \mathrm{d}[S]_t}{Y_t^2} =$$
$$\frac{C_t\sigma \mathrm{d}B_t + C_t\mu \mathrm{d}t}{Y_t} - \frac{1}{2}\frac{C_t^2\sigma^2 \mathrm{d}t}{Y_t^2} = \sigma X_t \mathrm{d}B_t + (\mu X_t - \frac{1}{2}\sigma^2 X_t^2)\mathrm{d}t = \sigma X_t \mathrm{d}B_t + f(X_t)\mathrm{d}t,$$

where we defined

$$X_t := \frac{C_t}{Y_t}, \qquad f(x) := \mu x - \frac{1}{2}\sigma^2 x^2.$$

We want to maximize the drift coefficient $f(X_t)$ by appropriately choosing X_t. The maximum of the function $f(x)$ is attained at

$$x^* = \frac{\mu}{\sigma^2}, \qquad \text{and} \qquad f(x^*) = \frac{1}{2}\frac{\mu^2}{\sigma^2},$$

thus we have $f(X_t) \le f(x^*) = \frac{1}{2}\frac{\mu^2}{\sigma^2}$ for any admissible trading strategy.
Our previous calculations give

$$\log\left(\frac{Y_T}{Y_0}\right) = \log(Y_T) - \log(Y_0) = \int_0^T \mathrm{d}\log(Y_t) = \int_0^T \sigma X_t \mathrm{d}B_t + \int_0^T f(X_t)\mathrm{d}t \le$$
$$\int_0^T \sigma X_t \mathrm{d}B_t + \int_0^T \frac{1}{2}\frac{\mu^2}{\sigma^2}\mathrm{d}t = \int_0^T \sigma X_t \mathrm{d}B_t + \frac{1}{2}\frac{\mu^2}{\sigma^2}T.$$

Now the expectation of $\int_0^T \sigma X_t \mathrm{d}B_t$ is zero (since it is an Itô integral w.r.t. Brownian motion), thus taking the expectation of both sides of our last calculation gives

$$\mathbb{E}\left[\log\left(\frac{Y_T}{Y_0}\right)\right] \le \frac{1}{2}\frac{\mu^2}{\sigma^2}T.$$

Here equality can be achieved if we choose $X_t \equiv x^* = \frac{\mu}{\sigma^2}$, i.e.,

$$C_t = \frac{\mu}{\sigma^2}Y_t \quad \text{for all} \quad 0 \le t \le T.$$

This is the trading strategy that achieves the maximal expected rate of return. Note that if $\mu > \sigma^2$ (i.e., if the upward „drift" μ of (S_t) is „stronger" than the „risk" quantified by σ^2) then $C_t > Y_t$, so you constantly have to be in debt in order to achieve the optimal rate of return.

5. Let $h : \mathbb{R}^2 \to \mathbb{R}$ be harmonic (i.e., $\Delta h \equiv 0$). Use Itō calculus, the optional stopping theorem and the rotation invariance of 2-dimensional Brownian motion (proved in Homework 6.1) to show

$$h(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} h\left(x_0 + r\cos(\varphi),\, y_0 + r\sin(\varphi)\right) \mathrm{d}\varphi$$

Solution. Let $\underline{x}_0 = (x_0, y_0)$. Let us consider a 2-dimensional Brownian motion $\underline{B}_t = (B_1(t), B_2(t))$ with starting point $\underline{x}_0$. Let D denote the disc with center $\underline{x}_0$ and radius r. Let τ denote the exit time of $(\underline{B}_t)$ from D, i.e.,

$$\tau = \min\{\, t \,:\, \|\underline{B}_t - \underline{x}_0\| = r \,\}.$$

Now $M_t = h(\underline{B}_t)$ is a martingale because $\Delta h \equiv 0$ (see lecture 23). Since harmonic functions are continuous, h is bounded on the compact set D, thus $(M_{t\wedge\tau})$ is a bounded martingale, so we can apply the optional stopping theorem to infer

$$h(x_0, y_0) = M_0 = \mathbb{E}(M_0) = \mathbb{E}(M_\tau) = \mathbb{E}(h(\underline{B}_\tau))$$

Now the law of $\underline{B}_\tau$ is a probability measure on ∂D, the circle of radius r around $\underline{x}_0$. This probability measure is invariant under rotations of the circle, because the law of the two-dimensional Brownian motion $(\underline{B}_t)$ is invariant under rotations of the plane around $\underline{x}_0$ and of course the domain D is also invariant under such rotations, so the value of τ remains the same for the rotated Brownian motion, and it is only the location where it reaches ∂D that gets rotated. Now there is only one probability measure on ∂D which is invariant under rotations, the uniform measure (this is very simple special case of the uniqueness of *Haar measure*), therefore

$$\mathbb{E}(h(\underline{B}_\tau)) = \frac{1}{2\pi} \int_0^{2\pi} h\left(x_0 + r\cos(\varphi),\, y_0 + r\sin(\varphi)\right) \mathrm{d}\varphi.$$