Stoch. Anal. HW assignment 8. Solutions

1. If $\underline{x} = (x, y) \in \mathbb{R}^2$, denote by $\|\underline{x}\| = \sqrt{x^2 + y^2}$ the Euclidean norm of $\underline{x}$. Let $\underline{B}_t = (B_1(t), B_2(t))$ denote a 2-dimensional Brownian motion starting from $\underline{x}_0 \neq \underline{0}$. Define $T_x = \inf\{\, t \,:\, \|\underline{B}_t\| = x \,\}$.

   The goal of this exercise is to show that

   - $(\underline{B}_t)$ always visits the ball of radius $a > 0$ around the origin but
   - $(\underline{B}_t)$ never hits the origin.

   (a) Calculate the stochastic differential $\mathrm{d}M_t$ of $M_t = \ln(\|\underline{B}_t\|)$.

   (b) Use the optional stopping theorem to calculate $\mathbb{P}[T_a < T_b]$ for any $0 < a < \|\underline{x}_0\| < b < +\infty$.

   (c) Show that $\mathbb{P}(T_a < +\infty) = 1$ for any $a > 0$. *Hint:* $b \to \infty$.

   (d) Show that $\mathbb{P}(T_0 < +\infty) = 0$. *Hint:* First $a \to 0$, then $b \to \infty$.

   Solution.

   (a) Let $f(\underline{x}) = \ln(\sqrt{x^2 + y^2}) = \frac{1}{2}\ln(x^2 + y^2)$. Then

   $$f_x = \frac{x}{\|\underline{x}\|^2}, \quad f_y = \frac{y}{\|\underline{x}\|^2}, \quad f_{xx} = \frac{\|\underline{x}\|^2 - 2x^2}{\|\underline{x}\|^4}, \quad f_{yy} = \frac{\|\underline{y}\|^2 - 2y^2}{\|\underline{x}\|^4}, \quad \Delta f = f_{xx} + f_{yy} = 0 \text{ if } \underline{x} \neq \underline{0}$$

   Now we apply the multivariate Itô formula:

   $$\mathrm{d}M_t = \mathrm{d}f(\underline{B}_t) = f_x(\underline{B}_t)\mathrm{d}B_1(t) + f_y(\underline{B}_t)\mathrm{d}B_2(t) + \Delta f(\underline{B}_t)\mathrm{d}t = \frac{B_1(t)}{\|\underline{B}_t\|^2}\mathrm{d}B_1(t) + \frac{B_2(t)}{\|\underline{B}_t\|^2}\mathrm{d}B_2(t).$$

   Thus the coefficient of $\mathrm{d}t$ in $\mathrm{d}M_t$ is zero.

   (b) Define $\tau = T_a \wedge T_b$, the first time when the Euclidean norm of our 2-dimensional Brownian motion reaches either a or b. The stopped process $M_{t\wedge\tau}$ is a bounded martingale, so we can apply the optional stopping theorem:

   $$\ln(\|\underline{x}_0\|) = \mathbb{E}(M_0) = \mathbb{E}(M_\tau) = \mathbb{P}(\tau = T_a)\ln(a) + \mathbb{P}(\tau = T_b)\ln(b) = \\ \mathbb{P}[T_a < T_b]\ln(a) + (1 - \mathbb{P}[T_a < T_b])\ln(b).$$

   Rearranging this we obtain

   $$\mathbb{P}[T_a < T_b] = \frac{\ln(b) - \ln(\|\underline{x}_0\|)}{\ln(b) - \ln(a)}.$$

   (c) Since $t \mapsto \underline{B}_t$ is a continuous function, $\|\underline{B}_t\|$ cannot go to infinity in finite time. Therefore we have $\lim_{b\to\infty} T_b = +\infty$. Also note that if $\|\underline{x}_0\| < b_1 \leq b_2$ then $T_{b_1} \leq T_{b_2}$, thus $\{T_a < T_{b_1}\} \subseteq \{T_a < T_{b_2}\}$ and $\{T_a < T_b\} \nearrow \{T_a < +\infty\}$ as $b \nearrow \infty$, so by the continuity of measure we have

   $$\mathbb{P}(T_a < +\infty) = \lim_{b\to\infty} \mathbb{P}(T_a < T_b) = \lim_{b\to\infty} \frac{\ln(b) - \ln(\|\underline{x}_0\|)}{\ln(b) - \ln(a)} = 1.$$

   (d) We have $\{T_0 < T_b\} \subseteq \{T_a < T_b\}$ for any $0 < a < \|\underline{x}_0\| < b < +\infty$, thus

   $$\mathbb{P}(T_0 < T_b) \leq \inf_{a>0} \mathbb{P}(T_a < T_b) = \lim_{a\to 0_+} \frac{\ln(b) - \ln(\|\underline{x}_0\|)}{\ln(b) - \ln(a)} = 0.$$

   Now $\{T_0 < T_b\} \nearrow \{T_0 < +\infty\}$ as $b \nearrow \infty$, so by the continuity of measure we have

   $$\mathbb{P}(T_0 < +\infty) = \lim_{b\to\infty} \mathbb{P}(T_0 < T_b) = 0.$$

Remark. Recall that if $Y_t = \|\underline{B}_t\|$ then (Y_t) is a two dimensional Bessel process, see lecture 26. We have just proved the $d = 2$ case (i.e., the critical case) of the recurrence/transience phase transition theorem of Bessel processes (see lecture 29) .

2. Let (X_t) be a Brownian motion with constant upward drift, i.e., the solution of the SDE

$$\mathrm{d}X_t = \mu \mathrm{d}t + \sigma \mathrm{d}B_t, \qquad X_0 = x_0 \in \mathbb{R}, \qquad \mu > 0,\ \sigma > 0.$$

(a) Find a non-constant function $f : \mathbb{R} \to \mathbb{R}$ such that $f(X_t)$ is a martingale.

(b) Use the above martingale and the optional stopping theorem to calculate $\mathbb{P}(T_a < T_b)$ where

$$a \le x_0 \le b, \qquad T_a = \min\{t \ :\ X_t = a\}, \qquad T_b = \min\{t \ :\ X_t = b\}$$

(c) Find a function $g : \mathbb{R} \to \mathbb{R}$ such that $g(X_t) - t$ is a martingale and calculate $\mathbb{E}(\tau)$, where $\tau = T_a \wedge T_b$.

Solution.

(a) First let us apply Ito's formula for $f(X_t)$:

$$\begin{aligned}\mathrm{d}f(X_t) = f'(X_t)\mathrm{d}X_t + \frac{1}{2}f''(X_t)\mathrm{d}[X]_t &= f'(X_t)(\mu \mathrm{d}t + \sigma \mathrm{d}B_t) + \frac{1}{2}f''(X_t)\sigma^2 \mathrm{d}t \\ &= \left(f'(X_t)\mu + \frac{\sigma^2}{2}f''(X_t)\right)\mathrm{d}t + f'(X_t)\sigma \mathrm{d}B_t.\end{aligned}$$

$f(X_t)$ can only be a martingale if $f'(X_t)\mu + \frac{\sigma^2}{2}f''(X_t) = 0$. To find such a function f, we have to solve the following differential equation: $f'(x)\mu + \frac{\sigma^2}{2}f''(x) = 0$. Let $v(x) = f'(x)$, so we solve $v(x)\mu + \frac{\sigma^2}{2}v'(x) = 0$. The solution is

$$v(x) = c_1 \cdot e^{-\frac{2\mu}{\sigma^2}x}.$$

Because of this, $f(x) = c_2 \cdot e^{-\frac{2\mu}{\sigma^2}x} + c_3$. The particular choice of $c_2 \neq 0$ and $c_3 \in \mathbb{R}$ doesn't matter for the rest of our calculations, so let us choose $c_2 = 1$ and $c_3 = 0$ for the sake of simplicity. Then $f(X_t) = e^{-\frac{2\mu}{\sigma^2}X_t}$ is a martingale.

(b) By the optional stopping theorem

$$\mathbb{E}(f(X_\tau)) = \mathbb{E}(f(X_0)) = f(X_0) = f(x_0).$$

On the other hand

$$\mathbb{E}(f(X_\tau)) = \mathbb{P}(T_a < T_b)f(a) + (1 - \mathbb{P}(T_a < T_b))f(b).$$

Combining these two we get

$$\mathbb{P}(T_a < T_b) = \frac{f(x_0) - f(b)}{f(a) - f(b)} = \frac{e^{-\frac{2\mu}{\sigma^2}x_0} - e^{-\frac{2\mu}{\sigma^2}b}}{e^{-\frac{2\mu}{\sigma^2}a} - e^{-\frac{2\mu}{\sigma^2}b}}.$$

Note that this result is the continuous analogue of HW2.3(b).

(c) We have $X_t = x_0 + \mu t + \sigma B_t$, so if we let $g(x) = \frac{x}{\mu}$ then $g(X_t) - t = \frac{x_0}{\mu} + \frac{\sigma}{\mu}B_t$ is obviously a martingale. Let us use the optional stopping theorem with the martingale $g(X_t) - t$ and the stopping time $\tau = T_a \wedge T_b$. This way,

$$\mathbb{E}(g(X_\tau) - \tau) = \mathbb{E}(g(X_0) - 0) = g(x_0),$$

so

$$\mathbb{E}(\tau) = \frac{1}{\mu}\left(\mathbb{E}(X_\tau) - x_0\right) = \frac{1}{\mu}\left(\frac{e^{-\frac{2\mu}{\sigma^2}x_0} - e^{-\frac{2\mu}{\sigma^2}b}}{e^{-\frac{2\mu}{\sigma^2}a} - e^{-\frac{2\mu}{\sigma^2}b}}a + \frac{e^{-\frac{2\mu}{\sigma^2}a} - e^{-\frac{2\mu}{\sigma^2}x_0}}{e^{-\frac{2\mu}{\sigma^2}a} - e^{-\frac{2\mu}{\sigma^2}b}}b - x_0\right)$$

Remark: One can check that if we let $\sigma = 1$ and $\mu \to 0$ in the above formula then we obtain

$$\mathbb{E}(\tau) = (b - x_0)(x_0 - a),$$

the expected exit time from $[a, b]$ of a standard Brownian motion started from x_0 with no drift.

3. Solve the following stochastic differential equations:

   (a) Find (Z_t) such that $\mathrm{d}Z_t = B_t Z_t \mathrm{d}t + B_t Z_t \mathrm{d}B_t$ with initial condition $Z_0 = 1$.

   (b) Find (Y_t) such that $\mathrm{d}Y_t = Y_t \mathrm{d}t + B_t \mathrm{d}B_t$ with initial condition $Y_0 = 1$.

Solution. These equations are all linear stochastic differential equations, see lecture 27.

(a) Observe that (Z_t) is the stochastic exponential (see lecture 24) of the process (G_t) whose stochastic differential is $\mathrm{d}G_t = B_t \mathrm{d}t + B_t \mathrm{d}B_t$. Thus $G_t - G_0 = \int_0^t B_s \mathrm{d}s + \int_0^t B_s \mathrm{d}B_s$. We have $[G]_t = \int_0^t B_s^2 \mathrm{d}s$. Thus by the definition of the stochastic exponential we have

$$Z_t = \exp\left(G_t - G_0 - \frac{1}{2}[G]_t\right) = \exp\left(\int_0^t B_s \mathrm{d}s + \int_0^t B_s \mathrm{d}B_s - \frac{1}{2}\int_0^t B_s^2 \mathrm{d}s\right)$$

We know the value of $\int_0^t B_s \mathrm{d}B_s$ from lecture 13, therefore we can further simplify our solution so that it does not contain Ito integrals, only Riemann integrals.

$$Z_t = \exp\left(\int_0^t \left(B_s - \frac{B_s^2}{2}\right) \mathrm{d}s + \frac{1}{2}B^2(t) - \frac{1}{2}t\right).$$

(b) We could use the general solution formula for inhomogeneous linear stochastic differential equations (see lecture 27), but let us do it by hand instead. Let us first solve the corresponding homogeneous equation: $\mathrm{d}\widetilde{Y}_t = \widetilde{Y}_t \mathrm{d}t$. This is actually an ODE: $\frac{\mathrm{d}\widetilde{Y}_t}{\mathrm{d}t} = \widetilde{Y}_t$. Thus $\widetilde{Y}_t = c \cdot e^t$.
Now we can solve the original inhomogeneous linear equation. Let us use the method of variation of constants, i.e., we look for the solution of the inhomogeneous equation in the form of $Y_t = C_t e^t$.

$$\mathrm{d}Y_t = \mathrm{d}(C_t e^t) = \mathrm{d}C_t e^t + C_t \mathrm{d}e^t + 0 = \mathrm{d}C_t e^t + C_t e^t \mathrm{d}t = e^t \mathrm{d}C_t + Y_t \mathrm{d}t.$$

We want $\mathrm{d}Y_t = Y_t \mathrm{d}t + B_t \mathrm{d}B_t$, so we need :

$$\begin{aligned} e^t \mathrm{d}C_t &= B_t \mathrm{d}B_t, \\ \mathrm{d}C_t &= e^{-t} B_t \mathrm{d}B_t, \\ C_t &= C_0 + \int_0^t e^{-s} B_s \mathrm{d}B_s. \end{aligned}$$

Now $1 = Y_0 = C_0 e^0$, thus $C_0 = 1$. Hence

$$Y_t = e^t + e^t \int_0^t e^{-s} B_s \mathrm{d}B_s.$$

4. The O.-U. process is defined by the formula

$$X_t = X_0 e^{-\alpha t} + \sigma e^{-\alpha t} \int_0^t e^{\alpha s} \mathrm{d}B_s.$$

(a) Manipulate the above formula to show that for any $s, t \in \mathbb{R}_+$ we have

$$X_{s+t} = X_s e^{-\alpha t} + \sigma e^{-\alpha t} \int_0^t e^{\alpha v} \mathrm{d}\widetilde{B}_v,$$

where $\widetilde{B}(v) = B(s+v) - B(s)$ for any $v \geq 0$, thus $(\widetilde{B}_v)$ is also a standard Brownian motion which is independent from $\mathcal{F}_s$.

(b) Show that (X_t) is a time-homogeneous Markov process (see lecture 10) and identify the corresponding transition density function $p_t(x, y)$.
Hint1: It suffices to show $\mathbb{P}(X_{s+t} \in [y, y + \mathrm{d}y] \,|\, \mathcal{F}_s) = p_t(X_s, y)\mathrm{d}y$ in order to solve (b).
Hint2: The O.-U. process is a Gaussian process, so if we fix t and x then $f(y) = p_t(x, y)$ will be the probability density function of a certain normal distribution.

Solution.

(a) For any $s, t \in \mathbb{R}_+$ we have

$$X_{s+t} = X_0 e^{-\alpha(s+t)} + \sigma e^{-\alpha(s+t)} \int_0^{s+t} e^{\alpha u} \mathrm{d}B_u =$$
$$\left(X_0 e^{-\alpha s} + \sigma e^{-\alpha s} \int_0^s e^{\alpha u} \mathrm{d}B_u \right) e^{-\alpha t} + \sigma e^{-\alpha(s+t)} \int_s^{s+t} e^{\alpha u} \mathrm{d}B_u =$$
$$X_s e^{-\alpha t} + \sigma e^{-\alpha t} \int_s^{s+t} e^{\alpha(u-s)} \mathrm{d}B_u = X_s e^{-\alpha t} + \sigma e^{-\alpha t} \int_0^t e^{\alpha v} \mathrm{d}\widetilde{B}_v,$$

where $(\widetilde{B}_v)_{v \geq 0}$ defined by $\widetilde{B}(v) = B(s+v) - B(s)$ is also a Brownian motion, independent from $\mathcal{F}_s$.

(b) First note that $\sigma e^{-\alpha t} \int_0^t e^{\alpha v} \mathrm{d}\widetilde{B}_v$ is normally distributed random variable with mean zero (because it is a constant multiplied by the integral of a deterministic function with respect to Brownian motion, see lecture 13). The variance is

$$\mathrm{Var}\left(\sigma e^{-\alpha t} \int_0^t e^{\alpha v} \mathrm{d}\widetilde{B}_v \right) = \sigma^2 e^{-2\alpha t} \int_0^t (e^{\alpha v})^2 \mathrm{d}v = \sigma^2 e^{-2\alpha t} \left(\frac{e^{2\alpha t} - 1}{2\alpha} \right) = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}).$$

Now X_s is $\mathcal{F}_s$-measurable and $\sigma e^{-\alpha t} \int_0^t e^{\alpha v} \mathrm{d}\widetilde{B}_v$ is independent of $\mathcal{F}_s$, therefore by part (a) the conditional distribution of X_{t+s} given $\mathcal{F}_s$ is normal $\mathcal{N}(\widetilde{\mu}, \widetilde{\sigma}^2)$, where

$$\widetilde{\mu} = X_s e^{-\alpha t}, \qquad \widetilde{\sigma}^2 = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}).$$

This means that the conditional probability density function of X_{s+t} given $\mathcal{F}_s$ is

$$\frac{\mathbb{P}(X_{s+t} \in [y, y + \mathrm{d}y] \,|\, \mathcal{F}_s)}{\mathrm{d}y} = \frac{1}{\widetilde{\sigma}\sqrt{2\pi}} \exp\left(-\frac{(y - \widetilde{\mu})^2)}{2\widetilde{\sigma}^2} \right) =$$
$$\frac{1}{\sigma\sqrt{\frac{\pi}{\alpha}(1 - e^{-2\alpha t})}} \exp\left(-\frac{(y - X_s e^{-\alpha t})^2}{\frac{\sigma^2}{\alpha}(1 - e^{-2\alpha t})} \right),$$

therefore

$$p_t(x, y) = \frac{1}{\sigma\sqrt{\frac{\pi}{\alpha}(1 - e^{-2\alpha t})}} \exp\left(-\frac{(y - x e^{-\alpha t})^2}{\frac{\sigma^2}{\alpha}(1 - e^{-2\alpha t})} \right).$$

Remark. Note that $\widetilde{\mu} \to 0$ and $\widetilde{\sigma} \to \frac{\sigma^2}{2\alpha}$ as $t \to \infty$, thus the stationary distribution of the O.-U. process is $\mathcal{N}(0, \frac{\sigma^2}{2\alpha})$. If we set $X_0 \sim \mathcal{N}(0, \frac{\sigma^2}{2\alpha})$, then (X_t) is a stationary Markov process. Now if we choose $\alpha = \beta$ and $\sigma = \sqrt{2\beta}$ then the stationary distribution is $\mathcal{N}(0, 1)$, which is in agreement with our calculations performed on lecture 13: $e^{-\beta t} X_0 + \sqrt{2\beta} \int_0^t e^{\beta \cdot (u-t)} \mathrm{d}B_u$ is a stationary process.