Stoch. Anal. HW assignment 9 Solutions

1. Find (X_t) such that
$$\mathrm{d}X_t = X_t \mathrm{d}B_t + X_t \mathrm{d}t + B_t \mathrm{d}B_t, \qquad X_0 = 1.$$

Solution. We use the solution formula of general linear stochastic differential equations, see lecture 27.
$\mathrm{d}X_t = X_t(\mathrm{d}B_t + \mathrm{d}t) + B_t \mathrm{d}B_t$.

First we calculate the stochastic exponential U_t, i.e., the solution of the corresponding homogeneous linear equation, i.e., a process which satisfies $U_0 = 1$ and $\mathrm{d}U_t = U_t(\mathrm{d}B_t + \mathrm{d}t)$. We obtain
$$U_t = \exp\left(B_t + \frac{1}{2}t\right).$$

Then $X_t = U_t C_t$, where
$$C_t = 1 + \int_0^t \frac{B_s}{U_s} \mathrm{d}B_s - \int_0^t \frac{B_s}{U_s} \mathrm{d}s.$$

2. Let $(u_t), (v_t)$ be left-continuous, adapted and almost surely bounded. Define

$$X_t := B_t + \int_0^t u_s \, \mathrm{d}s, \qquad M_t := \exp\left(\int_0^t v_s \, \mathrm{d}B_s - \frac{1}{2} \int_0^t v_s^2 \, \mathrm{d}s \right), \qquad Y_t := X_t M_t.$$

Given (u_t), how to choose (v_t) if you want both (M_t) and (Y_t) to be martingales?

Solution: Let us define the auxiliary process (V_t) by $V_t := \int_0^t v_s \, \mathrm{d}B_s$, $t \geq 0$.
In differential form, we have $\mathrm{d}V_t = v_t \, \mathrm{d}B_t$ with $V_0 = 0$.
Note that (M_t) is the stochastic exponential of (V_t) (see lecture 24). In particular, we have $\mathrm{d}M_t = M_t \, \mathrm{d}V_t$.
Thus $\mathrm{d}M_t = M_t \, \mathrm{d}V_t = M_t \, v_t \, \mathrm{d}B_t + 0 \, \mathrm{d}t$ (i.e., the drift term vanishes), thus (M_t) a martingale.
We have $\mathrm{d}X_t = \mathrm{d}B_t + u_t \, \mathrm{d}t$, thus we may use the stochastic product rule to calculate

$$\begin{aligned} \mathrm{d}Y_t = \mathrm{d}(X_t M_t) = X_t \, \mathrm{d}M_t + M_t \, \mathrm{d}X_t + \mathrm{d}\,[X, M]_t = \\ X_t M_t \, v_t \, \mathrm{d}B_t + M_t(\mathrm{d}B_t + u_t \, \mathrm{d}t) + M_t v_t \, \mathrm{d}t = M_t(X_t v_t + 1) \, \mathrm{d}B_t + M_t(u_t + v_t) \, \mathrm{d}t. \end{aligned}$$

If we want the $\mathrm{d}t$ term to vanish, we need to choose $v_t = -u_t$. This choice makes (Y_t) a martingale.

Remark: If we want to be precise, we also want to check that in the Itô integral representations

$$M_t = \int_0^t M_s \, v_s \, \mathrm{d}B_s, \qquad Y_t = \int_0^t M_s (X_s v_s + 1) \, \mathrm{d}B_s$$

the adapted processes that we integrate w.r.t. our Brownian motion have a finite $\| \cdot \|_*$-norm (c.f. lecture 11), because we can only guarantee the martingale property of Itô integrals w.r.t. B.M. under this condition, see lecture 12. In other words, for any $T \geq 0$ we want to check

$$\int_0^T \mathbb{E}\left[M_t^2 \, v_t^2 \right] \, \mathrm{d}t < +\infty, \qquad \int_0^T \mathbb{E}\left[M_t^2 (X_t v_t + 1)^2 \right] \, \mathrm{d}t < +\infty. \tag{1}$$

There is a theorem (that we did not learn about) called *Novikov's condition* which states that if (σ_t) is an adapted process and $\mathbb{E}\left[\exp\left(\frac{1}{2} \int_0^T \sigma_t^2 \, \mathrm{d}t \right) \right] < +\infty$ then $\left(\exp\left(\int_0^t \sigma_s \, \mathrm{d}B_s - \frac{1}{2} \int_0^t \sigma_s^2 \, \mathrm{d}s \right) \right)_{0 \leq t \leq T}$ is a martingale.
We assumed that $\mathbb{P}(\max_{0 \leq t \leq T} |u_t| \leq K) = 1$ and thus $\mathbb{P}(\max_{0 \leq t \leq T} |v_t| \leq K) = 1$ hold for some $K \in \mathbb{R}_+$.
Novikov's condition implies that $(U_t)_{0 \leq t \leq T}$ is a martingale, where $U_t = \exp\left(\int_0^t 4 v_s \, \mathrm{d}B_s - \frac{1}{2} \int_0^t (4 v_s)^2 \, \mathrm{d}s \right)$.
Thus $\mathbb{E}(U_t) = 1$ for all $0 \leq t \leq T$, which implies $\mathbb{E}\left[M_t^4 \right] \leq e^{Ct}$ for some constant $C = C(K) \in \mathbb{R}_+$.
Now $\mathbb{E}\left[M_t^2 (X_t v_t + 1)^2 \right]^2 \leq \mathbb{E}\left[M_t^4 \right] \mathbb{E}\left[(X_t v_t + 1)^4 \right]$ by Cauchy-Schwartz and finally it is easy to bound $\mathbb{E}\left[(X_t v_t + 1)^4 \right] \leq C' = C'(T)$ for all $t \in [0, T]$ using that $\mathbb{E}[B_t^4] \leq C'' t^2$ and $\mathbb{P}(\max_{0 \leq t \leq T} |u_t| \leq K) = 1$.
Putting these together we obtain the desired bounds (1).

3. We say that $(W_t)_{t\geq 0}$ is a Brownian motion *for* a filtration $(\mathcal{F}_t)_{t\geq 0}$ if

   (i) (W_t) is a standard Brownian motion,

   (ii) (W_t) is adapted to $(\mathcal{F}_t)$ and

   (iii) for any $0 \leq s \leq t$ the increment $W_t - W_s$ is independent of $\mathcal{F}_s$.

   Note that if (W_t) satisfies (i) and $(\mathcal{F}_t)$ is the natural filtration of (W_t) then (ii) and (iii) both hold. Also note that (i)&(ii)&(iii) together imply that (W_t) is a martingale with respect to $(\mathcal{F}_t)$. Throughout this course we tacitly assumed that our Brownian motions and filtrations satisfy (ii)&(iii). The goal of this exercise is to construct an example for which (i)&(ii) hold, but (iii) does not hold.

   Let (B_t) denote a standard Brownian motion and let $(\mathcal{F}_t)$ denote its natural filtration. Let us define

$$W_0 := 0, \qquad W_t := B_t - \int_0^t \frac{B_u}{u}\,\mathrm{d}u, \qquad t > 0. \tag{2}$$

   (a) Show that the above integral „makes sense" by showing that $\int_s^t \frac{B_u}{u}\,\mathrm{d}u$ converges in L_2 as $s \to 0_+$.
   Hint: Use integration by parts and Itō isometry (among other things).

   (b) Show that (W_t) is a standard Brownian motion.
   Hint: Observe that (W_t) is a Gaussian process...

   (c) Show that (W_t) satisfies (ii) but not (iii).
   Hint: (W_t) is an Itō process, but is it a martingale w.r.t. $(\mathcal{F}_t)$?

Solution:

(a) Let $0 < s \leq t$. We have

$$\int_s^t \frac{B_u}{u}\,\mathrm{d}u = \ln(t)B_t - \ln(s)B_s - \int_s^t \ln(u)\,\mathrm{d}B_u \tag{3}$$

using integration by parts (the mutual variation term vanishes). It is enough to show that the right-hand side of (3) converges in L_2 as $s \to 0_+$.
First we show that $\ln(s)B_s$ converges to zero in L_2 as $s \to 0_+$. Indeed:

$$\mathbb{E}[(\ln(s)B_s)^2] = \mathrm{Var}(\ln(s)B_s) = \ln^2(s)\mathrm{Var}(B_s) = \ln^2(s)\cdot s \to 0 \quad \text{as} \quad s \to 0_+.$$

Next we show that $\int_s^t \ln(u)\,\mathrm{d}B_u$ converges to $\int_0^t \ln(u)\,\mathrm{d}B_u$ in L_2 as $s \to 0_+$. Indeed: observe that we have $\int_0^t \ln^2(u)\,\mathrm{d}u < +\infty$ and thus

$$\mathbb{E}\left[\left(\int_0^t \ln(u)\,\mathrm{d}B_u - \int_s^t \ln(u)\,\mathrm{d}B_u\right)^2\right] = \mathbb{E}\left[\left(\int_0^s \ln(u)\,\mathrm{d}B_u\right)^2\right] = \int_0^s \ln^2(u)\,\mathrm{d}u \to 0, \quad s \to 0_+$$

by the dominated convergence theorem. Putting these together with (3) we obtain that

$$W_t = B_t - \int_0^t \frac{B_u}{u}\,\mathrm{d}u = B_t - \ln(t)B_t + \int_0^t \ln(u)\,\mathrm{d}B_u = \int_0^t (1 + \ln(u) - \ln(t))\,\mathrm{d}B_u. \tag{4}$$

(b) (W_t) is a Gaussian process because it is obtained from Brownian motion using linear operations. We have $\mathbb{E}(W_t) = 0$. Let us also calculate the autocovariance function of (W_t): if $0 < s \leq t$ then

$$\mathrm{Cov}(W_s, W_t) \overset{(*)}{=} \int_0^s (1 + \ln(u) - \ln(t))(1 + \ln(u) - \ln(s))\,\mathrm{d}u \overset{(**)}{=} s,$$

where $(*)$ is the covariance formula from lecture 13 and $(**)$ was calculated by Wolfram Alpha. Thus (W_t) is a Brownian motion by one of the equivalent characterizations learnt on lecture 7.

(c) We see from (2) that (W_t) is an Itō process driven by (B_t), thus (ii) holds (i.e., (W_t) is adapted to the natural filtration $(\mathcal{F}_t)$ of (B_t)), but (iii) cannot hold, because (i)&(ii)&(iii) together would imply that (W_t) is a martingale w.r.t. $(\mathcal{F}_t)$, but this is clearly not the case since the differential form of (2) is $\mathrm{d}W_t = \mathrm{d}B_t - \frac{B_t}{t}\,\mathrm{d}t$, the coefficient of $\mathrm{d}t$ does not vanish.

4. Let (X_t) be the solution of the SDE

$$\mathrm{d}X_t = \alpha X_t \,\mathrm{d}t + \sigma X_t^{\beta} \,\mathrm{d}B_t, \qquad X_0 = x_0, \tag{5}$$

where $\alpha, \beta, \sigma, x_0 \in \mathbb{R}$.

(a) Compute $\mathbb{E}(X_t)$. *Hint:* Write the SDE in integral form and take the expectation.

(b) Compute $\mathrm{Var}(X_t)$ when $\beta = 1/2$. *Hint:* Calculate $\mathrm{d}X_t^2$, rewrite in integral form, take expectation.

Solution:

(a) The drift term (see lecture 16) of (5) is $\mathbb{E}(\mathrm{d}X_t \,|\, \mathcal{F}_t) = \alpha X_t \,\mathrm{d}t$. Taking the expectation of both sides, using the tower property and the linearity of expectation we obtain

$$\mathrm{d}\mathbb{E}(X_t) = \mathbb{E}(\mathrm{d}X_t) = \mathbb{E}(\alpha X_t \,\mathrm{d}t) = \alpha \mathbb{E}(X_t)\mathrm{d}t.$$

Dividing both sides by $\mathrm{d}t$ we obtain the differential equation $\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}(X_t) = \alpha \mathbb{E}(X_t)$ with initial condition $\mathbb{E}(X_0) = x_0$, the solution of which is

$$\mathbb{E}(X_t) = x_0 e^{\alpha t}.$$

(b) Using Itō's formula for Itō processes we obtain

$$\mathrm{d}X_t^2 = 2X_t \,\mathrm{d}X_t + \frac{1}{2} 2 \,\mathrm{d}[X]_t \overset{(5)}{=} 2X_t(\alpha X_t \,\mathrm{d}t + \sigma X_t^{\beta} \,\mathrm{d}B_t) + \sigma^2 X_t^{2\beta} \,\mathrm{d}t$$

Taking the conditional expectation of both sides w.r.t. $\mathcal{F}_t$ and using that $\beta = 1/2$ we obtain

$$\mathbb{E}(\mathrm{d}X_t^2 \,|\, \mathcal{F}_t) = (2\alpha X_t^2 + \sigma^2 X_t) \,\mathrm{d}t.$$

Taking the expectation of both sides, using the tower property and linearity of expectation as well as $\mathbb{E}(X_t) = x_0 e^{\alpha t}$, finally dividing by $\mathrm{d}t$ we obtain the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}(X_t^2) = 2\alpha \mathbb{E}(X_t^2) + \sigma^2 x_0 e^{\alpha t}, \qquad \mathbb{E}(X_0^2) = x_0^2. \tag{6}$$

This is an inhomogeneous linear ODE, so first we solve the corresponding homogenized linear ODE, the general solution of which is $Ce^{2\alpha t}$, so we try to find the solution of the inhomogeneous linear ODE in the form $\mathbb{E}(X_t^2) = C_t e^{2\alpha t}$. Comparing the derivative of this expression with the r.h.s. of (6) we obtain that we must have $C_t' e^{2\alpha t} = \sigma^2 x_0 e^{\alpha t}$. Rearranging this, integrating it and observing that $C_0 = x_0^2$ must hold, we obtain that $C_t = \frac{1}{\alpha}\sigma^2 x_0 (1 - e^{-\alpha t}) + x_0^2$ and thus

$$\mathbb{E}(X_t^2) = C_t e^{2\alpha t} = \frac{1}{\alpha}\sigma^2 x_0 e^{2\alpha t}(1 - e^{-\alpha t}) + x_0^2 e^{2\alpha t}.$$

Finally, we obtain the desired formula for the variance of X_t:

$$\mathrm{Var}(X_t) = \mathbb{E}(X_t^2) - \mathbb{E}^2(X_t) = \frac{1}{\alpha}\sigma^2 x_0 e^{\alpha t}(e^{\alpha t} - 1).$$