First Midterm Exam - May 28, 2025, Stochastic Analysis

Family name _____ **Given name** _____

Signature _____ **Neptun Code** _____

No calculators or electronic devices are allowed. One formula sheet with 15 formulas is OK.

1. (7 points) Let (B_t) denote standard Brownian motion. Let

$$X_t = \int_0^t e^{3s}\,\mathrm{d}B_s.$$

Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ denote a monotone increasing continuous deterministic function satisfying $f(0) = 0$. How to choose f if I want the continuous-time process $(X_{f(t)})_{t\geq 0}$ to have the same law as (B_t)?

Hint: Use the equivalent characterization of Brownian motion as a Gaussian process with a given autocovariance structure.

2. (8 points) Let (B_t) denote the standard Brownian motion. Let $0 \leq \mu < +\infty$. Let us define $X_t := B_t + \mu t$. We call (X_t) the Brownian motion with upward drift μ.

Let $\tau = \min\{\, t \;:\; X_t = x \,\}$ denote the hitting time of level $x \in \mathbb{R}_+$. You may assume without proof that $\mathbb{P}(\tau < +\infty) = 1$.

   (a) For any $\alpha \in \mathbb{R}$ find a constant $\beta \in \mathbb{R}$ such that (M_t) is a martingale, where $M_t := \exp(\alpha X_t - \beta t)$.
   Hint: $\exp\left(\lambda B_t - \frac{\lambda^2}{2} t\right)$ is a martingale for any choice of $\lambda \in \mathbb{R}$.

   (b) Apply the optional stopping theorem to calculate $\mathbb{E}\left(e^{-\lambda\tau}\right)$ for any $\lambda \in \mathbb{R}_+$.
   Hint: You can use the following continuous-time form of the optional stopping theorem: if (M_t) is a martingale, τ is a stopping time satisfying $\mathbb{P}(\tau < +\infty) = 1$ and if there exists a constant $C \in \mathbb{R}_+$ such that $\mathbb{P}(|M_{t\wedge\tau}| \leq C) = 1$ for any $t \geq 0$ then we have $\mathbb{E}(M_\tau) = \mathbb{E}(M_0)$.