Stoch. Anal. Exam, June 4, 2025

Info: Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises (and sub-exercises) with a horizontal line. No calculators or electronic devices are allowed. One hand-written two-sided A4-sized formula sheet with 30 formulas is allowed. You have 90 minutes to complete this exam.

1. Let (B_t) denote a standard Brownian motion and $(\mathcal{F}_t)$ its natural filtration.

   (a) (3 points) Define the notion of a simple predictable process.

   (b) (3 points) Define the notion of the Itô integral of this simple predictable process with respect to Brownian motion.

   (c) (7 points) State and prove Itô's isometry in the case when the integrand is a simple predictable process.

   Instruction: if you use results about discrete-time stochastic processes that we have learnt earlier than proving Itô's isometry is class (e.g. homework exercises) then please clearly state the results that you use and explain why they can be applied here.

2. (12 points) Let $(B(t))$ denote a standard Brownian motion. Use the reflection principle to show that for any $x > 0$ and $\varepsilon > 0$ we have

$$\mathbb{P}(B(t) \geq x - \varepsilon \mid \min_{0 \leq s \leq t} B(s) \geq -\varepsilon) = \frac{\Phi\left(\frac{\varepsilon - x}{\sqrt{t}}\right) - \Phi\left(\frac{-\varepsilon - x}{\sqrt{t}}\right)}{2\Phi\left(\frac{\varepsilon}{\sqrt{t}}\right) - 1}.$$

3. (a) (5 points) Given an Itô process (X_t), find the Itô process (U_t) for which $U_t = 1 + \int_0^t U_s \, \mathrm{d}X_s$ holds for all $t \geq 0$. *Instruction:* Please present the derivation of your solution in detail.

   (b) (8 points) Given the Itô processes (X_t), (Y_t), find the Itô process (V_t) for which

$$\mathrm{d}V_t = V_t \mathrm{d}X_t + \mathrm{d}Y_t, \qquad V_0 = v_0.$$

   Instruction: Please present the derivation of your solution in detail.

4. (12 points) State and prove Tanaka's formula (for Brownian motion).