Stoch. Anal. Exam, June 11, 2025

Info: Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises (and sub-exercises) with a horizontal line. No calculators or electronic devices are allowed. One hand-written two-sided A4-sized formula sheet with 30 formulas is allowed. You have 90 minutes to complete this exam.

1. This is an exercise on discrete-time stochastic processes. Let $(\mathcal{F}_n)_{n=0}^{\infty}$ denote a filtration.
   (a) (3 points) Define the following types of stochastic processes: adapted, predictable, martingale.
   (b) (7 points) Define the notion of the discrete stochastic integral of a predictable process with respect to a martingale, and prove that it is also a martingale.
   (c) (2 points) Briefly explain why the intuitive meaning of this theorem is: *You can't beat the system.*

2. (12 points) You are trading shares at the stock market. The value of one stock at time t is
$$S_t = \sigma B_t + \mu t,$$
where (B_t) is standard Brownian motion and $\mu > 0$ (this is a toy example so let's not worry about the fact that S_t can become negative). At time t you hold C_t shares. Denote by $Y_0 > 0$ your initial wealth and by Y_t your total wealth at time t. It is OK to go in debt (i.e., it is OK if $C_t > Y_t$). Let us assume that you can't predict the future, so that (C_t) is left-continuous and adapted to $(\mathcal{F}_t)$, where $\mathcal{F}_t = \sigma(S_u,\, 0 \le u \le t)$. We have learnt in class your net gain from trading shares at time t is
$$Y_t - Y_0 = \int_0^t C_u \,\mathrm{d}S_u.$$
You trade in stocks until time T. Your goal is to maximize your expected rate of return $\mathbb{E}\left[\log\left(\frac{Y_T}{Y_0}\right)\right]$. What is the maximal expected rate of return achievable and the trading strategy that achieves it?
Hint: Let $X_t := \frac{C_t}{Y_t}$ denote the fraction of your wealth that you keep in shares (noting that $X_t > 1$ means that you are in debt). First calculate the stochastic differential $\mathrm{d}\log(Y_t)$ and express the drift of $\log(Y_t)$ in terms of X_t. How to choose C_t if we want to maximize the drift? Switching back from differential form to integral form, you will be able to tell which trading strategy maximizes $\mathbb{E}\left[\log\left(\frac{Y_T}{Y_0}\right)\right]$.

3. (13 points) Let (B_t) denote standard Brownian motion. Show that the following three continuous-time processes (all three are defined for $0 \le t \le 1$) all have the same law:
   (a) The conditional distribution of $(B_t)_{0\le t\le 1}$ given $B_1 = 0$
   (b) The process $(X_t)_{0\le t\le 1}$ defined by $X_t := B_t - tB_1$
   (c) The solution $(Y_t)_{0\le t\le 1}$ of the SDE $\mathrm{d}Y_t = \frac{-Y_t}{1-t}\,\mathrm{d}t + \mathrm{d}B_t$ with $Y_0 = 0$ on the interval $t \in [0, 1)$.

4. (13 points) State and prove Girsanov's theorem.
Instruction: You can use without proof that if (u_t) is left-continuous, adapted, and Novikov's condition holds and if we define
$$X_t := B_t + \int_0^t u_s \,\mathrm{d}s, \qquad M_t := \exp\left(\int_0^t (-u_s)\,\mathrm{d}B_s - \frac{1}{2}\int_0^t u_s^2 \,\mathrm{d}s\right), \qquad Y_t := X_t M_t,$$
then (M_t) and (Y_t) are both martingales.