PROBABILITY MIDTERM 1., 2025.10.13.

$\boxed{\text{GROUP A}}$

$\boxed{1}$ a) $\dfrac{\binom{5}{2}\cdot\binom{15}{3}}{\binom{20}{5}}$

b) $1-\mathbb{P}(\text{ALL BALLS HAVE THE SAME COLOR}) =$

$$= 1 - \frac{4}{\binom{20}{5}}$$

c) $A_i := \{\text{WE PICK AT LEAST ONE BALL OF COLOR } i\}$

$$\mathbb{P}\left(\bigcap_{i=1}^{4} A_i\right) = 1-\mathbb{P}\left(\bigcup_{i=1}^{4} A_i^c\right) = ★$$

IF $I \subseteq \{1,2,3,4\}$ THEN $\mathbb{P}\left(\bigcap_{i\in I} A_i^c\right) = \dfrac{\binom{20-5\cdot|I|}{5}}{\binom{20}{5}}$

$$★ \overset{\boxed{\text{INCL.-EXCL.}}}{=\!=} 1-\sum_{r=1}^{4}(-1)^{r+1}\sum_{\substack{I\subseteq[4]\\|I|=r}}\frac{\binom{20-5\cdot r}{5}}{\binom{20}{5}} =$$

$$= 1-\sum_{r=1}^{4}(-1)^{r+1}\cdot\binom{4}{r}\cdot\frac{\binom{20-5\cdot r}{5}}{\binom{20}{5}} = \boxed{0.3225}$$

$$1-\binom{4}{1}\cdot\frac{\binom{15}{5}}{\binom{20}{5}}+\binom{4}{2}\cdot\frac{\binom{10}{5}}{\binom{20}{5}}-\binom{4}{3}\cdot\frac{\binom{5}{5}}{\binom{20}{5}}+0$$

$\boxed{2}$ a) POI(λ) BY BINOMIAL APPROXIMATION OF POI:

n FISH , p = PROB. THAT I CATCH THAT FISH

$\boxed{n \gg 1}$ $\boxed{p \ll 1}$ $\boxed{n \cdot p \approx 1}$ CATCH: BIN(n, p)

$n \cdot p = \lambda$

$e^{-\lambda} = \frac{1}{3} \implies \boxed{\lambda = \ln(3)}$

b) IF HURON: POI(μ), $e^{-\mu} = \frac{1}{2}$, $\mu = \ln(2)$

$\mathbb{P}$(CATCH TODAY ≥ 2) =

$$= \frac{1}{2} \cdot \left(1 - e^{-\lambda} - \lambda \cdot e^{-\lambda}\right) + \frac{1}{2} \cdot \left(1 - e^{-\mu} - \mu \cdot e^{-\mu}\right) = 0.227$$

c) $\mathbb{P}$(ERIE | CATCH = 2) =

$$\frac{\mathbb{P}(\text{ERIE AND CATCH} = 2)}{\mathbb{P}(\text{CATCH} = 2)} = \frac{\frac{1}{2} \cdot e^{-\lambda} \cdot \frac{\lambda^2}{2}}{\frac{1}{2} \cdot e^{-\lambda} \cdot \frac{\lambda^2}{2} + \frac{1}{2} e^{-\mu} \cdot \frac{\mu^2}{2}} =$$

$= p = 0.626$

d) $p \cdot \left(1 - e^{-\lambda}\right) + (1 - p) \cdot \left(1 - e^{-\mu}\right) = 0.604$

BONUS : $n+1$ PEOPLE BUY LOTTERY TICKETS

EACH OF THEM WIN WITH PROBABILITY p

INDEPENDENTLY OF EACH OTHER.

1 UNIT OF MONEY IS DISTRIBUTED
EVENLY AMONG THE WINNERS.

IF NOBODY WINS, NO MONEY IS DISTIBUTED.

M = EXPECTED TOTAL AMOUNT OF
MONEY THAT THE PLAYERS WIN

m = EXPECTED AMOUNT OF MONEY
THAT I WIN

$\boxed{M=(n+1)\cdot m}$ $\boxed{M = 1-(1-p)^{n+1}}$

IF I WIN THEN THE NUMBER OF
OTHER WINNERS IS $X \sim BIN(n,p)$

THUS $\boxed{m = p \cdot \mathbb{E}\left(\frac{1}{1+X}\right)}$ SINCE I SHARE THE PRICE WITH X OTHER WINNERS

THUS $\boxed{\mathbb{E}\left(\frac{1}{1+X}\right) = \frac{1-(1-p)^{n+1}}{(n+1)\cdot p}}$

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$\boxed{\text{GROUP B}}$

$\boxed{1}$ a) $\binom{5}{2} \cdot \left(\frac{1}{4}\right)^2 \cdot \left(\frac{3}{4}\right)^3$ | b) $1 - 4 \cdot \left(\frac{1}{4}\right)^5$

c) $A_i :=$ {WE PICK AT LEAST ONE BALL OF COLOR i}

$$\mathbb{P}\left(\bigcap_{i=1}^{4} A_i\right) = 1 - \mathbb{P}\left(\bigcup_{i=1}^{4} A_i^c\right) = \star$$

IF $I \subseteq \{1,2,3,4\}$ THEN $\mathbb{P}\left(\bigcap_{i \in I} A_i^c\right) = \left(\frac{4-|I|}{4}\right)^5$

$$\star \overset{\text{INCL.-EXCL.}}{=\!=} 1 - \sum_{r=1}^{4} (-1)^{r+1} \cdot \sum_{\substack{I \subseteq [4] \\ |I| = r}} \left(\frac{4-r}{4}\right)^5 =$$

$$1 - \sum_{r=1}^{4} (-1)^{r+1} \cdot \binom{4}{r} \cdot \left(\frac{4-r}{4}\right)^5 =$$

$$1 - \binom{4}{1} \cdot \left(\frac{3}{4}\right)^5 + \binom{4}{2} \cdot \left(\frac{2}{4}\right)^5 - \binom{4}{3} \cdot \left(\frac{1}{4}\right)^5 + 0 =$$

$$= \frac{15}{64} = 0.234$$

$\boxed{2}$ a) POI (3) BECAUSE OF POI APPROX. OF BINOMIAL:

n PIECES OF MEAT IN STEW

p = PROBABILITY THAT A PIECE OF MEAT ENDS UP ON MY PLATE

$\boxed{n \gg 1}$ $\boxed{p \ll 1}$ $\boxed{n \cdot p = 3}$ $\mathrm{BIN}(n, p) \approx \mathrm{POI}(3)$

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b) $\frac{2}{5} \cdot (1 - e^{-3}) + \frac{3}{5} \cdot (1 - e^{-1}) = 0.759$

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c) $\mathbb{P}(\text{GORD} \mid 3 \text{ MEAT}) = \frac{\mathbb{P}(\text{GORD AND } 3 \text{ MEAT})}{\mathbb{P}(3 \text{ MEAT})} =$

$$= \frac{\frac{2}{5} \cdot e^{-3} \cdot \frac{3^3}{3!}}{\frac{2}{5} \cdot e^{-3} \cdot \frac{3^3}{3!} + \frac{3}{5} \cdot e^{-1} \cdot \frac{1^3}{3!}} = p = 0.709$$

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d) $p \cdot (1 - e^{-3} - e^{-3} \cdot 3) + (1 - p) \cdot (1 - e^{-1} - e^{-1} \cdot 1)$

BONUS: $\mathbb{E}\left(\frac{e^X}{1+X}\right) = \sum_{k=0}^{\infty} \frac{e^k}{1+k} \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!} =$

$$= e^{-\lambda} \cdot \sum_{k=0}^{\infty} \frac{(\lambda \cdot e)^k}{(k+1)!} = e^{-\lambda} \cdot \frac{1}{\lambda \cdot e} \cdot \sum_{k=0}^{\infty} \frac{(\lambda \cdot e)^{k+1}}{(k+1)!} =$$

$$= e^{-\lambda} \cdot \frac{1}{\lambda \cdot e} \cdot \left(\exp(\lambda \cdot e) - 1\right)$$