NAME: .... NEPTUN CODE: .... MAJOR: ....

Probability Theory Exam, January 14, 2025

Working time: 100 min. Only simple, non-programmable calculators are allowed, standard normal distribution table on the other side. The achievable maximum score (with the Bonus exercise) is 110 points, but we consider 100 points as 100%.

T. 1. Let (X, Y) be jointly absolutely continuous random variables with joint density function $f(x, y)$.

(a) *(1+1+1 points)* Define the marginal density function $f_X(x)$, the conditional density function $f_{Y|X}(y|x)$, and the conditional expectation $\mathbb{E}(Y|X = x)$ using formulas.

(b) *(1+4 points)* State and prove the tower property of conditional expectation.

(c) *(2 points)* State the formula for the conditional variance ($\mathrm{Var}(Y) = \dots$).

(d) *(3+4 points)* Let $X \sim \mathrm{EXP}(\lambda)$ and given a fixed $X = x$, let Y have the distribution $\mathrm{UNI}[0, x]$. Compute the expected value and variance of Y. *Hint*: The variance of $\mathrm{UNI}[0, x]$ is $x^2/12$.

T. 2. (a) *(3+2+2 points)* Let $(X_1, \dots, X_n)$ be a general n-dimensional jointly normal random vector. Write down the joint density function of $(X_1, \dots, X_n)$, explaining the dimensions of the vectors or matrices in the formula, the covariance matrix, and the relationship between the matrix in the density function and the covariance matrix.

(b) *(5 points)* Let (X, Y) be jointly normal with $\mathrm{Cov}(X, Y) = 0$. Prove that X and Y are independent.

(c) *(5 points)* Give an example of a pair of random variables (X, Y) that are not independent but satisfy $\mathrm{Cov}(X, Y) = 0$.

T. 3. (a) *(2+1+4 points)* Define the (optimistic) geometric distribution with parameter p (both with a formula and an intuitive explanation) and compute its expected value.

(b) *(2+2 points)* Define the memoryless property and prove that the geometric distribution has this property.

(c) *(5 points)* Let X and Y be independent random variables with $\mathrm{GEO}(p)$ distribution. Compute the conditional probability $\mathbb{P}(X = 6 \mid X + Y = 15)$.

P. 1. After the holiday sales, Thomas bought 1000 light bulbs because they were on sale. The lifetime of each bulb follows an exponential distribution with an expected value of 3 months. If a bulb burns out, Thomas (or his descendants) immediately replace it with a new one.

(a) *(3 points)* What is the expected number and variance of bulb replacements in the first year?

(b) *(5 points)* Compute the distribution function of the time of the second bulb replacement. *Hint:* The time of the second bulb replacement is greater than t if and only if there are fewer than two replacements in the interval $[0, t]$.

(c) *(9 points)* Approximate the largest T such that the bulb supply lasts for T years with a 90% probability. (A standard normal distribution table is on the back.)

P. 2. Determine the density functions of X and Y if

(a) *(8 points)* ξ is uniformly distributed on the interval $[-3, 2]$, and $X = \xi^2$,

(b) *(8 points)* ϑ follows an $\mathrm{EXP}(\lambda)$ distribution, and $Y = \sqrt{\vartheta}$.

Bonus *(10 points)* Let $X \sim \mathrm{EXP}(\lambda_1)$ and $Y \sim \mathrm{EXP}(\lambda_2)$ be two independent random variables. Compute the density function of $Z = X + Y$.

P. 3. Let X and Y be independent normally distributed random variables. X has mean 0 and variance 4, and Y has mean 0 and variance 9. Compute the probability $\mathbb{P}\left((X, Y) \in H\right)$ where

(a) *(8 points)* $H = \{(x, y) \in \mathbb{R}^2 : x \le -y, -1 \le y \le 1, x \ge -3\}$,

(b) *(9 points)* $H = \{(x, y) \in \mathbb{R}^2 : (\frac{x}{4})^2 + (\frac{y}{6})^2 \le 2\}$.

$\Phi(z)$

z	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000