PROBAB. THEO. MIDTERM 2. | 2025.11.24 | A

(1) a) $F(x) = 0$ IF $x < 0$, BUT IF $x \geq 0$ THEN

$$F(x) = C \cdot \int_0^x \frac{1}{(u+1)^3}\,du = C \cdot \left[\left(-\frac{1}{2}\right) \cdot \frac{1}{(u+1)^2}\right]_0^x =$$

$$= C \cdot \frac{1}{2} \cdot \left(1 - \frac{1}{(1+x)^2}\right) \overset{\boxed{C=2}}{=\!=} 1 - \frac{1}{(1+x)^2}$$

THUS $F(x) \overset{x}{\underset{\infty}{\to}} 1$

b) X = JUMP LENGTH $\quad \mathbb{E}(X) = \int_0^\infty 2 \cdot \frac{x}{(1+x)^3}\,dx =$

$$\int_0^\infty 2 \cdot \frac{x+1}{(1+x)^3}\,dx - \int_0^\infty 2 \cdot \frac{1}{(1+x)^3}\,dx =$$

$$2 \cdot \int_0^\infty \frac{1}{(x+1)^2}\,dx - 2 \cdot \int_0^\infty \frac{1}{(1+x)^3}\,dx = 2 - 1 = 1$$

$$\mathbb{E}(X^2) = \int_0^\infty 2 \cdot \frac{x^2}{(1+x)^3}\,dx \;\cup\!\!\cap\; \int_1^\infty \frac{1}{x}\,dx = +\infty$$

THUS $\operatorname{Var}(X) = +\infty$

c) X_1, X_2, X_3 INDEP. AND IDENTICALLY DISTRIBUTED

$Y := \max\{X_1, X_2, X_3\} \quad G(x) = \mathbb{P}(Y \leq x) =$

$$\mathbb{P}(X_1 \leq x, X_2 \leq x, X_3 \leq x) = \mathbb{P}(X \leq x)^3 = F(x)^3 =$$

$$= \left(1 - \frac{1}{(1+x)^2}\right)^3 \text{ IF } x \geq 0, \text{ OTHERWISE } G(x) = 0$$

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P.D.F. OF LONGEST JUMP:

$$g(x) = G'(x) = \frac{d}{dx}\left(F(x)\right)^3 = 3\cdot\left(F(x)\right)^2 \cdot F'(x) =$$

$$= 3\cdot\left(1-\frac{1}{(1+x)^2}\right)^2 \cdot 2 \cdot \frac{1}{(1+x)^3}$$

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d) $P(Y \geq 5) = 1 - G(5) =$

$$= 1 - \left(1-\frac{1}{6^2}\right)^3 = 0.081$$

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(2) $X_n \sim BIN(n, 0.55)$

X_n = NUMBER OF STUDENTS WHO PASS

WANT: $0.03 \geq P\left(\frac{X_n}{n} \leq 0.5\right) = P\left(X_n \leq 0.5 \cdot n\right) =$

$$= P\left(\frac{X_n - (0.55)\cdot n}{\sqrt{(0.55)\cdot(0.45)\cdot n}} \leq \frac{(0.5)\cdot n - (0.55)\cdot n}{\sqrt{(0.55)\cdot(0.45)\cdot n}}\right) \overset{\text{DE MOIVRE}}{\approx}$$

$$\approx \Phi\left(\frac{-(0.05)}{\sqrt{(0.55)\cdot(0.45)}}\cdot\sqrt{n}\right) = 1 - \Phi\left((0.1)\cdot\sqrt{n}\right)$$

WANT: $\Phi\left((0.1)\cdot\sqrt{n}\right) \geq 0.97$ WANT: $\frac{\sqrt{n}}{10} \geq 1.89$

WANT: $n \geq 357.2$

AT LEAST 358 STUDENTS

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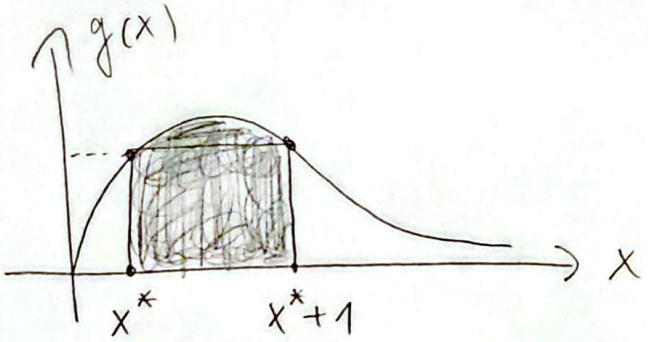
BONUS: UNIT OF MEASUREMENT: MINUTE

Y = TIME OF SECOND CALL

$Y = X_1 + X_2$, WHERE X_1, X_2 ARE INDEPENDENT $\exp(\frac{1}{5})$ RANDOM VARIABLES.

P.D.F. OF X_1 IS $\frac{1}{5} \cdot e^{-x/5} \cdot \mathbb{I}[0<x] = f(x)$

P.D.F. OF Y IS $g(x) = (f * f)(x) = \frac{1}{25} \cdot x \cdot e^{-x/5}$



I WANT TO MAXIMIZE $\int_{x^*}^{x^*+1} g(x)\,dx$, SO

CLEARLY, I WANT $g(x^*) = g(x^*+1)$

$$\frac{1}{25} \cdot x \cdot e^{-x/5} = \frac{1}{25} \cdot (x+1) \cdot e^{-(x+1)/5}$$

$$x = (x+1) \cdot e^{-1/5} = x \cdot e^{-1/5} + e^{-1/5}$$

$$x^* = \frac{e^{-1/5}}{1-e^{-1/5}} = \frac{1}{e^{1/5}-1} = 0.4516 = x^*$$

THE BEST INTERVAL OF LENGTH 1 IS $[x^*, x^*+1]$

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