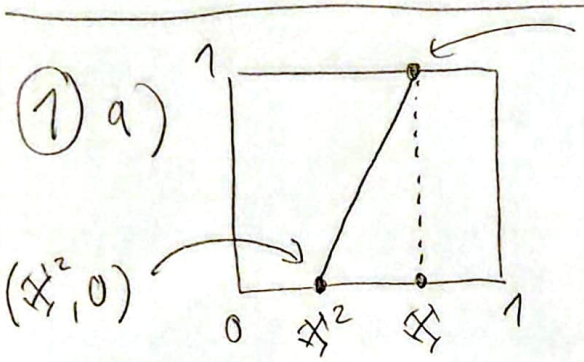
PROBAB. 2ND MIDTERM (RETAKE) | 2025.12.10.

① a)



SLOPE: $\frac{1-0}{X-X^2} = \frac{1}{X\cdot(1-X)}$

C.D.F. OF SLOPE:

$$F(x) = \mathbb{P}\left(\frac{1}{X\cdot(1-X)} < x\right) = \mathbb{P}\left(\frac{1}{\frac{1}{4}-(X-\frac{1}{2})^2} < x\right) =$$

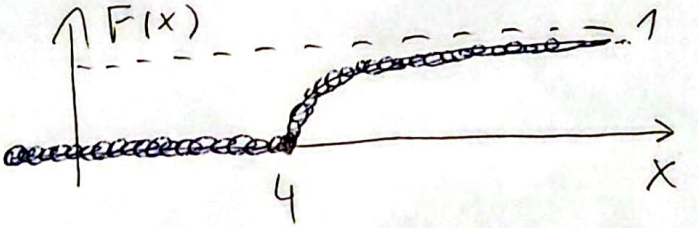
$$= \mathbb{P}\left(\frac{1}{x} < \frac{1}{4}-(X-\frac{1}{2})^2\right) = \mathbb{P}\left((X-\frac{1}{2})^2 < \frac{1}{4}-\frac{1}{x}\right) =$$

$$= \begin{cases} 0 \text{ IF } x<4 \\ (\bigstar) \text{ IF } x\geq 4 \end{cases} \qquad (\bigstar) = \mathbb{P}\left(|X-\tfrac{1}{2}| < \sqrt{\tfrac{1}{4}-\tfrac{1}{x}}\right) =$$

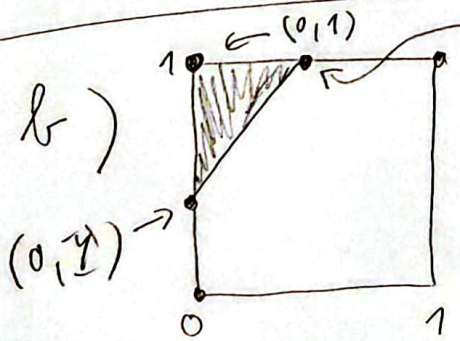
$$= \mathbb{P}\left(\frac{1}{2}-\sqrt{\frac{1}{4}-\frac{1}{x}} < X < \frac{1}{2}+\sqrt{\frac{1}{4}-\frac{1}{x}}\right) =$$

$$= 2\cdot\sqrt{\frac{1}{4}-\frac{1}{x}}$$

$$= \sqrt{1-4/x}$$



b)



AREA $= (1-Y)\cdot X\cdot\frac{1}{2}$

$$\mathbb{E}(\text{AREA}) = \frac{1}{2}\mathbb{E}((1-Y)\cdot X) =$$

$$= \frac{1}{2}\cdot\mathbb{E}(1-Y)\cdot\mathbb{E}(X) = \text{☺} \quad \boxed{\text{INDEP}}$$

$$\mathbb{E}(X) = \frac{0+1}{2} = \frac{1}{2}$$

$$\mathbb{E}(1-Y) = 1-\mathbb{E}(Y) = 1-\frac{1}{2} = \frac{1}{2}$$

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(iv) $= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$

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b) ALTERNATIVE SOLUTION:

$$\mathbb{E}(\text{AREA}) \overset{\text{LOTUS}}{=} \int_0^1 \int_0^1 \frac{1}{2} \cdot (1-y) \cdot x \cdot f(x,y)\, dx\, dy =$$

$$= \int_0^1 \int_0^1 \frac{1}{2} \cdot (1-y) \cdot x \, dx\, dy = \frac{1}{2} \cdot \left(\int_0^1 x\, dx \right) \cdot \left(\int_0^1 (1-y)\, dy \right) =$$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

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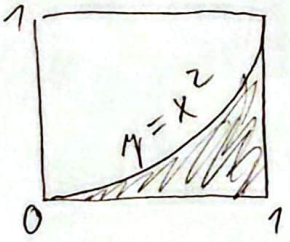
c)



$$\mathbb{P}(Y < X^2) = \frac{\text{SHADED AREA}}{\text{TOTAL AREA}} =$$

$$= \text{SHADED AREA} = \int_0^1 x^2\, dx = \frac{1}{3}$$

(ii) X' = NUMBER OF SUCCESSES OUT OF n ATTEMPTS

MY ESTIMATE OF p IS $\frac{X'}{n}$

NOTE: $X' \sim \text{BIN}(n, p)$

$$\mathbb{P}\left(\left|\frac{X'}{n} - p\right| < 0.01\right) = \mathbb{P}\left(|X' - n \cdot p| < \frac{n}{100}\right)$$

$$= \mathbb{P}\left(\left|\frac{X' - n \cdot p}{\sqrt{n \cdot p \cdot (1-p)}}\right| < \frac{\sqrt{n}}{100 \cdot \sqrt{p \cdot (1-p)}}\right) = \text{(ii)}$$

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(ii) $\overset{\text{DE MOIVRE}}{\approx} \Phi\left(\frac{\sqrt{n}}{100\cdot\sqrt{p\cdot(1-p)}}\right) - \Phi\left(-\frac{\sqrt{n}}{100\cdot\sqrt{p\cdot(1-p)}}\right) =$

$= 2\cdot\Phi\left(\frac{\sqrt{n}}{100\cdot\sqrt{p\cdot(1-p)}}\right) - 1 \overset{\text{WANT}}{=\!=} 0.9$

WANT: $\Phi\left(\frac{\sqrt{n}}{100\cdot\sqrt{p\cdot(1-p)}}\right) = 0.95$

FROM TABLE: $\Phi^{-1}(0.9505) = 1.65$

THUS $\frac{\sqrt{n}}{100\cdot\sqrt{p\cdot(1-p)}} \geq 1.65$

THUS $n \geq (1.65)^2\cdot 10^4\cdot p\cdot(1-p)$

$\max_{0\leq p\leq 1} p\cdot(1-p) = \frac{1}{4}$ (USING HINT OF EXERCISE 1(a))

THUS $n = \lceil (1.65)^2\cdot 10^4\cdot\frac{1}{4}\rceil = 6807$

SO n SHOULD BE AT LEAST 6807

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(2) b) FIRST SHOT: $X \sim N(m, \sigma^2)$

SECOND SHOT: $Y \sim N(-m, \sigma^2)$

$Z_1 = X + Y$ = WHERE IT LANDS

$Z_1 \sim N(0, 2\cdot\sigma^2)$

$$\frac{1}{2} = P(\text{SUCCESS}) = P\left(Z_1 \in \left[-\frac{1}{2}, \frac{1}{2}\right]\right) =$$

$$= P\left(\frac{Z_1}{\sqrt{2}\cdot\sigma} \in \left[-\frac{1}{2\cdot\sqrt{2}\cdot\sigma}, \frac{1}{2\cdot\sqrt{2}\cdot\sigma}\right]\right) =$$

$$\Phi\left(\frac{1}{2\cdot\sqrt{2}\cdot\sigma}\right) - \Phi\left(\frac{-1}{2\cdot\sqrt{2}\cdot\sigma}\right) =$$

$$2\cdot\Phi\left(\frac{1}{2\cdot\sqrt{2}\cdot\sigma}\right) - 1$$

$$\Phi\left(\frac{1}{2\cdot\sqrt{2}\cdot\sigma}\right) = \frac{3}{4} = 0.75$$

FROM TABLE: $\Phi^{-1}(0.75) \approx 0.675$

THUS $\frac{1}{2\cdot\sqrt{2}\cdot\sigma} \approx 0.675$

$$\boxed{\sigma \approx 0.5237}$$

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BONUS: IF $Z_n = \min\{X_1, \dots, X_n\}$

WHERE $X_1, \dots, X_n$ ARE I.I.D. $EXP(2)$

THEN $Z_n \sim EXP(2n)$

PROOF: $P(Z_n \geq x) = P(X_i \geq x)^n = (e^{-2x})^n =$

$= e^{-2nx}$ ← INDEED THE SURVIVAL FUNCTION OF $EXP(2n)$

CLAIM: $W = \max\{X_1, X_2, X_3\}$ HAS THE SAME DISTRIBUTION AS $Y_3 + Y_2 + Y_1$, WHERE $Y_i \sim EXP(2i)$ AND Y_1, Y_2, Y_3 ARE INDEP.

PROOF: X_1, X_2, X_3 ARE THE LIFETIMES OF THREE RADIOACTIVE ATOMS.

$Y_3 = \min\{X_1, X_2, X_3\}$ IS THE TIME IT TAKE FOR THE FIRST ATOM TO DECAY.

THEN, BY MEMORYLESS PROPERTY, WE ARE LEFT WITH TWO ATOMS.

THE GAP BETWEEN THE FIRST AND SECOND DECAY IS Y_2 AND THE GAP BETWEEN THE SECOND AND THE LAST ONE IS Y_1

$$E(W) = E(Y_1) + E(Y_2) + E(Y_3) = \frac{1}{2} + \frac{1}{4} + \frac{1}{6}$$

$$Var(W) = \sum_{i=1}^{3} Var(Y_i) = \sum_{i=1}^{3} \left(\frac{1}{2i}\right)^2$$

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