Name: .... NEPTUN code: .... Major: ....

Probability Theory 1 exam, January 23rd, 2025

Working hours: 100 min. Non-programable calculator without internet connection can be used.
Maximal amount of points (with the Bonus): 110 points, but 100 points are already considered as 100%.

The. 1. (a) *(2+2 points)* Define the binomial distribution $\mathrm{BIN}(n,p)$ based on its descriptive meaning and derive from this the probabilities $\mathbb{P}(X=k)$ for the possible values of k if $X \sim \mathrm{BIN}(n,p)$.

(b) *(3+4 points)* Determine the expected value and variance of $\mathrm{BIN}(n,p)$.

(c) *(2+4 points)* State and prove the Poisson approximation theorem.

Bonus *(10 points)* Let $X \sim \mathrm{BIN}(n,p)$. Give a closed formula for $\mathbb{E}(\frac{1}{X+1})$.

The. 2. (a) *(6 points)* Define the probability density function $f_X(x)$ of an absolutely continuous random variable X. Give and prove the defining properties of f_X.

(b) *(10 points)* Let Y be a non-negative absolutely continuous random variable. Show that $\mathbb{E}(Y) = \int_0^\infty \mathbb{P}(Y > y)dy$.

The. 3. Let (X,Y) be jointly absolutely continuous random variables with joint density function $f(x,y)$.

(a) *(1+1+1 points)* Define the marginal density f_X, the conditional density $f_{Y|X}(y|x)$ and the conditional expectation $\mathbb{E}(Y|X)$.

(b) *(1 points)* State the tower rule.

(c) *(1 points)* Using the conditional expectation, define the conditional variance $\mathbb{D}^2(X|Y)$.

(d) *(1+5 points)* State and prove the formula for conditional variance.

(e) *(2+4 points)* Denote the number of customers of a shop on a day by N, which is random. Let ξ_i be the spending of the ith customer, which is also random. We know that $\mathbb{E}(N) = 200$, $\mathbb{D}^2(N) = 81$, moreover, $\mathbb{E}(\xi_i) = 5000$ and $\mathbb{D}^2(\xi_i) = 10000$ for every i. Suppose that $N, \xi_1, \xi_2, \dots$ are independent. Denote Y the total income of the shop on a day. Find the expected value and variance of Y.

Prac. 1. In the Bonferroni factory, there are two production lines: every third car is made on the first line, while the rest is made on the second. Every Bonferroni car stalls at the ignition with a small probability (independently at each ignition). However, this probability depends on the production line where the car was made. The probability that a car made on the first production line stalls at least once in a year is $1/4$, while the probability of the same is $1/2$ on the second production line.

(a) (8 points) Lewis is an average customer who bought a Bonfferroni car and he wants to use it for two years. What is the probability that Lewis' car stalls exactly two times during this two years?

(b) (8 points) Max is an another average customer who is already using his Bonfferroni car for a year, and during this year his car did not stall at all. What is the probability that Max's car won't stall in the next year?

Prac. 2. *(8+9 points)* We throw an unbiased dice 30 times. Let X be the number of occasions when an even number is followed by an odd. What is the expected value and variance of X?

Prac. 3. *(17 points)* Given 48 random real numbers, we round them one by one to the closest integer. Assume that the rounding errors for different numbers are independent, identically distributed, and uniform on the interval $[-\frac{1}{2}, \frac{1}{2}]$. Denote S the sum of these 48 numbers and denote T the sum of the integers that we obtain after rounding each of the summands. What is the probability that we get T by rounding S? Use normal approximation. (*Hint:* if $\xi \sim \mathrm{UNI}(a,b)$ then $\mathbb{D}^2(\xi) = \frac{(b-a)^2}{12}$.)

$\Phi(z)$

z	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000