VSZ1 VIZSGA | 2024. 12. 17.

GYAK 1 X = KINYÍLÓ LEPKE-CSAPDÁK SZÁMA

$X \sim BIN(5, 2/3)$ $\mathbb{E}(X) = 5 \cdot \frac{2}{3}$, $Var(X) = 5 \cdot \frac{2}{3} \cdot \frac{1}{3}$

MINDEN CSAPDÁBA 3/ÓRA RÁTÁJÚ POISSON-FOLYAMAT SZERINT ÉRKEZNEK A LEPKÉK

Y = BEFOGOTT LEPKÉK SZÁMA

HA $X = k$, AKKOR $Y \sim POI(k \cdot 3)$

$\mathbb{E}(Y | X = k) = 3 \cdot k$, $\mathbb{E}(Y|X) = 3 \cdot X$

$Var(Y | X = k) = 3 \cdot k$, $Var(Y|X) = 3 \cdot X$

$$\underbrace{\mathbb{E}(Y) = \mathbb{E}(\mathbb{E}(Y|X))}_{\text{TORONSZABÁLY:}} = \mathbb{E}(3 \cdot X) = 3 \cdot 5 \cdot \frac{2}{3} = 10$$

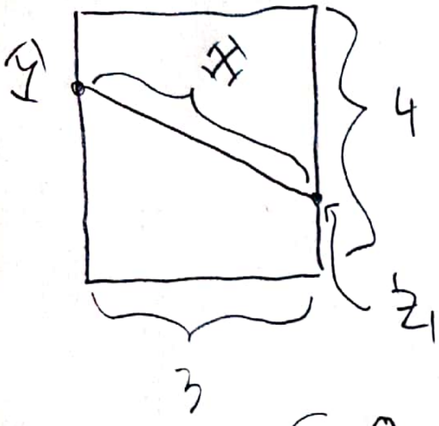
$$\underbrace{Var(Y) = \mathbb{E}(Var(Y|X)) + Var(\mathbb{E}(Y|X))}_{\text{FELTÉTELES SZÓRÁS}^{\square}\text{ FORMULA:}} =$$

$$= \mathbb{E}(3 \cdot X) + Var(3 \cdot X) =$$

$$= 3 \cdot \mathbb{E}(X) + 9 \cdot Var(X) =$$

$$= 3 \cdot 5 \cdot \frac{2}{3} + 9 \cdot 5 \cdot \frac{2}{3} \cdot \frac{1}{3} = 10 + 10 = 20$$

$\boxed{\text{GYAK 2}}$ Y_1, Z_1 F.A.E. UNI[0,4]



$$\boxed{X = \sqrt{(Y_1 - Z_1)^2 + 3^2}}$$

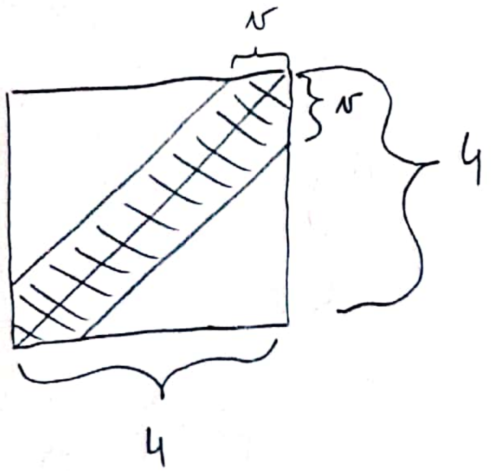
X ELOSZLÁSFÜGGVÉNYE: $F(x)$

$$F(x) = \begin{cases} 0, \text{ HA } x \le 3 \\ ★, \text{ HA } x \in (3,5) \\ 1, \text{ HA } x \ge 5 \end{cases}$$

$$★ = \mathbb{P}(X \le x) = \mathbb{P}\left((Y_1 - Z_1)^2 + 3^2 \le x^2\right) =$$

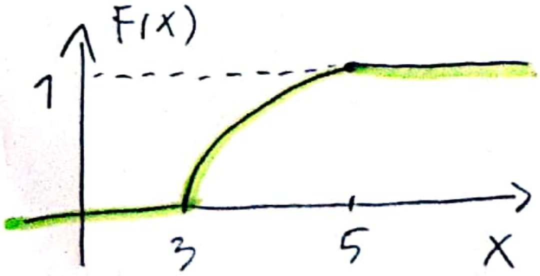
$$\mathbb{P}\left(|Y_1 - Z_1| \le \underbrace{\sqrt{x^2 - 9}}_{v}\right) = \mathbb{P}\left((Y_1, Z_1) \in \text{SATÍROZOTT}\right) =$$

$$= \frac{16 - (4 - v)^2}{16} =$$



$$= 1 - \left(1 - \frac{v}{4}\right)^2 =$$

$$= 1 - \left(1 - \frac{\sqrt{x^2 - 9}}{4}\right)^2$$



$$f(x) = F'(x) =$$

$$\frac{1}{8} \cdot x \cdot \left(\frac{4}{\sqrt{x^2 - 9}} - 1\right) \cdot \mathbb{1}[3 < x < 5]$$

2.OLDAL

$\boxed{\text{GYAK 3}}$ X, Y F.A.E. $N(0,3)$

$$X^* = \frac{X}{\sqrt{3}}, \quad Y^* = \frac{Y}{\sqrt{3}}$$ EKKOR (X^*, Y^*) 2-DIM STANDARD NORMÁLIS

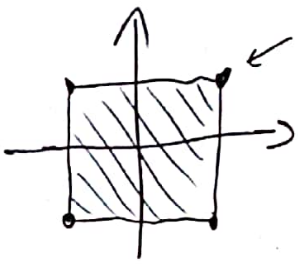
a) $$\mathbb{P}\big((X,Y) \in \text{NÉGYZET}\big) = \mathbb{P}\left((X^*,Y^*) \in \frac{1}{\sqrt{3}} \cdot \text{NÉGYZET}\right)$$

$$= \mathbb{P}\Big((X^*,Y^*) \in \underbrace{45^\circ\text{-AL ELFORGATOTT } \tfrac{1}{\sqrt{3}} \cdot \text{NÉGYZET}}\Big)$$

$$= \left(2 \cdot \Phi\left(\frac{1}{2} \cdot \frac{\sqrt{2}}{\sqrt{3}}\right) - 1\right)^2$$

$$= \big(2 \cdot \Phi(0.408) - 1\big)^2$$

$$= (2 \cdot 0.6591 - 1)^2 = 0.1012$$



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b) $$\mathbb{P}\left((X,Y) \in B\left(\tfrac{3}{2}\right)\right) = \mathbb{P}\left((X^*,Y^*) \in B\left(\tfrac{\sqrt{3}}{2}\right)\right) =$$

$$= \iint_{B\left(\frac{\sqrt{3}}{2}\right)} \frac{1}{2\pi} \cdot e^{-(x^2+y^2)/2} \, dx\, dy \stackrel{\text{POLÁR}}{=\!=} \int_0^{\sqrt{3}/2} \int_0^{2\pi} \frac{1}{2\pi} \cdot e^{-r^2/2} \cdot r \, d\varphi \, dr =$$

$$= \int_0^{\sqrt{3}/2} \left(-e^{-r^2/2}\right)' dr = 1 - e^{-\left(\frac{\sqrt{3}}{2}\right)^2/2} = 1 - e^{-3/8} =$$

$$= 0.3127$$

ELM 1

a) $(\Omega, \mathcal{F}, \mathbb{P})$

$\mathbb{P}(\emptyset)=0$, $\mathbb{P}(\Omega)=1$, ÉS HA

$A_1, A_2, A_3, \ldots \in \mathcal{F}$, $A_i \cap A_j = \emptyset$ HA $i \neq j$,

AKKOR $\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$

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b) $$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{\substack{I \subseteq [n] \\ I \neq \emptyset}} (-1)^{|I|+1} \cdot \mathbb{P}\left(\bigcap_{i \in I} A_i\right)$$

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c) $$\mathbb{P}(A \cup B) = \mathbb{P}(A \,\dot{\cup}\, (B \setminus A)) =$$
$$\mathbb{P}(A) + \mathbb{P}(B \setminus A) = \mathbb{P}(A) + \mathbb{P}(B \setminus (A \cap B)) =$$
$$= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

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d) $A_i = \{$ i-EDIK DIÁK MAGÁT HÚZTA $\}$

$$\mathbb{P}\left(\bigcap_{i=1}^{29} A_i^c\right) = 1 - \mathbb{P}\left(\bigcup_{i=1}^{29} A_i\right) = 1 - \sum_{\substack{I \subseteq [29] \\ I \neq \emptyset}} (-1)^{|I|+1} \cdot \frac{(29-|I|)!}{29!} =$$
$$= 1 + \sum_{k=1}^{29} \binom{29}{k} \cdot (-1)^k \cdot \frac{(29-k)!}{29!} = 1 + \sum_{k=1}^{29} (-1)^k \cdot \frac{1}{k!} =$$
$$= \sum_{k=0}^{29} (-1)^k \cdot \frac{1}{k!} \approx \frac{1}{e}$$

$\boxed{\text{ELM 2}}$ a) $\varphi(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}$

$\mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot \varphi(x)\, dx = 0$ (MERT PÁRATLAN FV-T INTEGRÁLTUNK)

$\mathrm{Var}(X) = \mathbb{E}(X^2) - 0^2 = \int_{-\infty}^{\infty} x^2 \cdot \varphi(x)\, dx = \int_{-\infty}^{\infty} x \cdot (-\varphi'(x))\, dx =$

PARCIÁLIS INTEGR.

$= \underbrace{\left[x \cdot (-\varphi(x))\right]_{-\infty}^{\infty}}_{= 0 - 0 = 0} - \underbrace{\int_{-\infty}^{\infty} 1 \cdot (-\varphi(x))\, dx}_{= \int_{-\infty}^{\infty} \varphi(x)\, dx = 1} = 1$

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b) $f(x) = \frac{1}{\sqrt{\pi}} \cdot \exp\left(-(x-1)^2\right) = \frac{1}{\sqrt{2\pi} \cdot \sqrt{\frac{1}{2}}} \cdot \exp\left(-\frac{(x-1)^2}{2 \cdot \frac{1}{2}}\right)$

TEHÁT $X \sim \mathcal{N}(1, \frac{1}{2})$, ÍGY $Y \sim \mathcal{N}(\mu, \sigma^2)$,

AHOL $\mu = \mathbb{E}(Y) = \mathbb{E}(\sqrt{2} \cdot (1 - X)) = 0$

$\sigma^2 = \mathrm{Var}(Y) = \mathrm{Var}(\sqrt{2} \cdot (1 - X)) = \sqrt{2}^2 \cdot \mathrm{Var}(X)$

$= 2 \cdot \frac{1}{2} = 1$

TEHÁT $Y \sim \mathcal{N}(0, 1)$, ÍGY $\mathrm{Var}(Y) = 1$ ✓

$\boxed{\text{ELM 3}}$ a) $\mathrm{Cov}(X,Y) = \mathbb{E}(X Y) - \mathbb{E}(X)\cdot\mathbb{E}(Y)$

$$\mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)}\cdot\sqrt{\mathrm{Var}(Y)}}$$

b) $\underline{\underline{C}} = (c_{i,j})_{i,j=1}^{n}$

$c_{i,j} = \mathrm{Cov}(X_i, X_j)$

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c) $\underline{v}^T = (v_1, \dots, v_n) \in \mathbb{R}^n$ KELL: $\underline{v}^T \underline{\underline{C}}\, \underline{v} \geq 0 \quad \forall \underline{v}$

BIZ: $\underline{v}^T \cdot \underline{\underline{C}} \cdot \underline{v} = \sum_{i,j=1}^{n} \mathrm{Cov}(X_i, X_j)\cdot v_i \cdot v_j \ \underline{\underline{\text{BILINEARITÁS}}}$

$= \mathrm{Cov}\left(\sum_{i=1}^{n} X_i \cdot v_i, \sum_{j=1}^{n} X_j \cdot v_j\right) = \mathrm{Var}\left(\sum_{i=1}^{n} v_i \cdot X_i\right) \geq 0$

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d) ÁLL: $|\mathrm{Cov}(X,Y)| \leq \sqrt{\mathrm{Var}(X)}\cdot\sqrt{\mathrm{Var}(Y)}$

BIZ: (X,Y) KOV.-MÁTRIXA POZITÍV SZEMIDEFINIT,

TEHÁT A DETERMINÁNSA NEMNEGATÍV:

$\mathrm{Var}(X)\cdot\mathrm{Var}(Y) - \mathrm{Cov}(X,Y)\cdot\mathrm{Cov}(X,Y) \geq 0$ ✓

BÓNUSZ: $\mathrm{Var}(X) = \frac{(9-6)^2}{12} = \frac{3}{4}$, ÍGY X ÉS Y KORRELÁCIÓJA:

$\mathrm{Corr}(X,Y) = \frac{3/2}{\sqrt{\frac{3}{4}}\cdot\sqrt{3}} = 1$, TEHÁT $Y = \alpha\cdot X + \beta$ ALAKÚ

TEHÁT $Y \sim \mathrm{UNI}[a,b]$, AHOL $\frac{a+b}{2} = 10$ ÉS $\frac{(a-b)^2}{12} = \sqrt{3}^2$

TEHÁT $Y \sim \mathrm{UNI}[7,13]$, ÍGY

$\mathbb{P}(Y \geq 9) = \frac{13-9}{13-7} = \frac{4}{6} = \frac{2}{3}$