Fractals and geometric measure theory
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Multifractal analysis
Measures, local dimension

Self-similar measures

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Multifractal analysis of self-similar measures with SSP

References
We write \( \mathcal{M} \) for the set of measures \( \mu \) satisfying:
- \( \mu \) is a Radon measure,
- \( \text{spt}(\mu) \) is compact,
- \( 0 < \mu(\mathbb{R}^d) < \infty \).

Let

\[
\mathcal{M}_1 := \{ \mu \in \mathcal{M} : \mu \text{ is a probability measure} \} .
\]  

(1)

Let \( A \subset \mathbb{R}^d \) be a Borel set. Further, we define

\[
\mathcal{M}(A) := \{ \mu \in \mathcal{M} : \text{spt}(\mu) \subset A \} ,
\]

\[
\mathcal{M}_1(A) := \{ \mu \in \mathcal{M}(A) : \mu(\mathbb{R}^d) = 1 \} .
\]
Hausdorff dimension of a measure

Let $\mu \in \mathcal{M}$. Recall: we have introduced the definition:

**Definition**

$$\dim_H(\mu) := \inf \left\{ \dim_H(A) : \mu(\mathbb{R}^d \setminus A) = 0 \right\}.$$  

Recall: we have proved the following theorem

**Theorem**

$$\dim_H(\mu) = \text{ess sup}_x \liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r}. \quad (2)$$

Roughly speaking, $\dim_H(\mu) = \delta$ if for a $\mu$-typical $x$ we have $\mu(B(x, r)) \approx r^\delta$ for small $r > 0$. 
Local dimension I

Let $\mu \in \mathcal{M}_1$. From now we denote the local dimension by $d_\mu(x)$ instead of $\dim_{\text{loc}}(\mu, x)$. That is

$$d_\mu(x) := \liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r},$$

$$\overline{d}_\mu(x) := \limsup_{r \to 0} \frac{\log \mu(B(x, r))}{\log r},$$

$$d_\mu(x) := \lim_{r \to 0} \frac{\log \mu(B(x, r))}{\log r},$$

The lower local dimension, upper local dimension, local dimension is defined by:

$$d_\mu(x), \overline{d}_\mu(x), d_\mu(x)$$ respectively.
What we have just proved it is a theorem due to Lai Sang Young:

Figure: Lai Sang Young
Theorem (L.S. Young)

Let $\Lambda \subset \mathbb{R}^d$ be measurable and $\mu(\Lambda) > 0$. Suppose that for every $x \in \Lambda$,

$$a \leq \liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} \leq b. \quad (3)$$

Then

$$a \leq \dim_H(\Lambda) \leq b. \quad (4)$$

Clearly, all limits remains unchanged if instead of $r \to 0$ we change to a sequence $r_n \downarrow 0$ satisfying

$$\lim_{n \to \infty} \frac{r_n}{r_{n+1}} = 1.$$
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For a probability vector

\[ \mathbf{p} = (p_1, \ldots, p_m). \]

we define the infinite product measure:

\[ \mathbf{p}^\mathbb{N} := (p_1, \ldots, p_m)^\mathbb{N}. \]

We are also given a self-similar IFS

\[ S = \{S_1, \ldots, S_m\} \]

on \( \mathbb{R}^d \) with contraction ratios

\[ 0 < r_i < 1. \]
Recall that the similarity dimension $s$ was defined as the solution of the equation

$$r_1^s \cdots + r_m^s = 1.$$  \hspace{1cm} (5)

Using the natural projection (coding) $\Pi$,

$$\Pi(i) := \lim_{n \to \infty} S_{i_1 \cdots i_n}(0),$$

we consider the push down measure of $p^\mathbb{N}$:

$$\nu := \Pi_* p^\mathbb{N},$$  \hspace{1cm} (6)
self-similar measures III

that is, for Borel $A \subset \mathbb{R}^d$:

$$\nu(A) := p^N(\Pi^{-1}(A)).$$

**Homework** Prove that for Borel $A \subset \mathbb{R}^d$

$$\nu(A) = \sum_{i=1}^{m} p_i \nu(S_i^{-1}(A)).$$ (7)

**Theorem**

*There is a unique measure $\mu \in \mathcal{M}_1$ satisfying (7).*
Idea of the proof

By (7): \( \text{spt}(\nu) \subset \Lambda \). We introduce the metric \( L(\mu, \eta) \) for \( \mu, \eta \in \mathcal{M}_1(\Lambda) \):

\[
L(\mu, \eta) := \sup \left\{ \mu(\phi) - \eta(\phi) \mid \phi : \Lambda \to \mathbb{R}, \text{ Lip}(\phi) \leq 1 \right\}.
\]

Further, consider the operator \( \mathcal{F} : \mathcal{M}_1(\Lambda) \to \mathcal{M}_1(\Lambda) \):

\[
(\mathcal{F}_\nu)(\phi) := \sum_{k=1}^{m} p_i \int \phi \circ S_i d\nu.
\]

Then

(a) The metric space \((\mathcal{M}_1(\Lambda), L)\) is complete.

(b) \( \mathcal{F} \) is a contraction on \((\mathcal{M}_1(\Lambda), L)\).

So, by Banach fixed point theorem we obtain that there is a unique fixed point of \( \mathcal{F} \). \( \square \)
Natural measure (the definition)

Let $s$ be the similarity dimension of the IFS $\mathcal{S}$. The important special case is:

$$\nu := \prod_*(p^\mathbb{N}) \text{ for } p = (r_1^s, \ldots, r_m^s).$$

The measure $\nu$ is called the natural measure on $\Lambda$. 

The natural measure I

Fact

Assume that the IFS $S = \{S_i(x) = r_i x + t_i\}_{i=1}^m$ satisfies the OSC. Then

$$d_\nu(x) \equiv s \text{ holds } \forall x \in \Lambda.$$ 

We remark that Young’s Theorem and this Fact implies that

$$\dim_H \nu = s. \quad (9)$$

Proof We give the proof in the special case when the Strong Separation Property holds that is we assume that the sets $\Lambda_i := S_i(\Lambda), \ i = 1, \ldots, m$ are pairwise disjoint.
The natural measure II

Without loss of generality we may assume that $|\Lambda| = 1$. Let

$$d := \min \text{dist}(\Lambda_i, \Lambda_j), \ i \neq j.$$  

Set $r_{\text{max}} := \max \{r_1, \ldots, r_m\}$. Fix an $\ell$ such that

$$r^\ell_{\text{max}} < d.$$  

Fix an arbitrary $x = \Pi(i)$ and $r > 0$. We define $n$ such that

$$r_{i_1 \ldots i_{n+\ell}} \leq r_{i_1 \ldots i_n}d \leq r < r_{i_1 \ldots i_{n-1}}d. \quad (10)$$  

Then

$$\Lambda_{i_1 \ldots i_{n+\ell}} \subset B(x, r) \cap \Lambda \subset \Lambda_{i_1 \ldots i_n}. \quad (11)$$
The natural measure III

Hence

$$r_{i_1...i_n+\ell}^s \leq \nu(B(x, r)) \leq r_{i_1...i_n}^s,$$  \hspace{1cm} (12)

Putting together (10) and (14) we obtain

$$\frac{\log r_{i_1...i_n}^s}{\log r_{i_1...i_n+\ell}} < \frac{\log \nu(B(x, r))}{\log r} \leq \frac{\log r_{i_1...i_n+\ell}^s}{\log r_{i_1...i_{n-1}}^s d}$$ \hspace{1cm} (13)

Now let $r \to 0$ to get the assertion of the Fact. \qed
Recall that we are given a self-similar IFS $S = \{S_1, \ldots, S_m\}$ with contractions $r := (r_1, \ldots, r_m)$ respectively. As always $\Lambda$ is the attractor and $\Sigma := \{1, \ldots, m\}^\mathbb{N}$ is the symbolic space. Further we write (as always) $\Pi : \Sigma \to \Lambda$,

$$\Pi(i) := \lim_{n \to \infty} S_{i_1, \ldots, i_n}(0), \quad i = (i_1, i_2, \ldots).$$
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Entropy, Lyapunov exponent

For a given probability vector \( \mathbf{p} = (p_1, \ldots, p_m) \) we consider the self-similar measure:

\[
\nu := \prod_* \mathbf{p}^\mathbb{N}.
\]

Set

\[
h_p := - \sum_{j=1}^{m} p_j \log p_j \quad \text{and} \quad \kappa_{p,r} := - \sum_{j=1}^{m} p_j \log r_j.
\]

We call

- \( h_p \) the entropy.
- \( \kappa_{p,r} \) the Lyapunov exponent.
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Local dimension of self-similar measures assuming OSC I

In what follows we always assume that the OSC holds.

Theorem

\[
d_{\nu}(x) = \lim_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} = \frac{h_p}{K_{p,r}} \text{ for } \nu\text{-a.e } x \in \Lambda. \quad (15)
\]
Local dimension of self-similar measures assuming OSC II

**Proof:** We give the proof for the case when the SSP holds. That is for

\[ d := \min \text{dist}(\Lambda_i, \Lambda_j), \quad i \neq j, \]

\[ d > 0. \] Like above, we set \( r_{\max} := \max \{ r_1, \ldots, r_m \} \) and we fix an \( \ell \) such that

\[ r_{\max}^\ell < d. \]

We obtained on slide 16 that

\[ r_{i_1 \ldots i_{n+\ell}} \leq \nu(B(x, r)) \leq r_{i_1 \ldots i_n}^s, \quad (16) \]
Local dimension of self-similar measures assuming OSC III

A similar argument as on slide 16 yields that for \( \mu \)-a.e. \( i \in \Sigma \)

\[
\lim_{r \to 0} \frac{\log \nu(B(x, r))}{\log r} = - \lim_{n \to \infty} \frac{1}{n} \left( \log p_{i_1} + \cdots + \log p_{i_n} \right) - \lim_{n \to \infty} \frac{1}{n} \left( \log r_{i_1} + \cdots + \log r_{i_n} \right) = \frac{h_p}{\kappa_{p,r}},
\]

(17)

where in the last step we used the LLN both in the nominator and denominator.
Remark

It follows from (11) that

\[ d_\nu(x) = \lim_{r \to 0} \frac{\log \nu(B(x,r))}{\log r} = \lim_{n \to \infty} \frac{\log p_{i_1 \ldots i_n}}{\log r_{i_1 \ldots i_n}}. \]  

That is whenever the limit \( \lim_{r \to 0} \frac{\log \nu(B(x,r))}{\log r} \) exists then the limit on the right hand side in (18) also exists and the two limits are the same. This is true even in the (\( \nu \)-atypical) case when this limit is not equal to \( \frac{h_p}{\kappa_{p,r}} \).
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In this section we consider a self-similar measure $\nu$ and study the size (Hausdorff dimension) of the set $K_\alpha$ where the local dimension $d_\nu(x) = \lim_{r \to 0} \frac{\log \nu(B(x,r))}{\log r}$ of the measure $\mu$ is equal to a given number $\alpha$. That is

$$K_\alpha := \{ x \in \Lambda : d_\nu(x) = \alpha \}.$$  \hspace{1cm} (19)

The object of our study is, the function

$$D : \alpha \mapsto \dim_H(K_\alpha)$$  \hspace{1cm} (20)

Clearly $K_\alpha = \emptyset$ if $\alpha \not\in [\alpha_1, \alpha_2]$, where

$$\alpha_{\text{min}} := \min_{1 \leq i \leq m} \frac{\log p_i}{\log r_i}, \quad \text{and} \quad \alpha_{\text{max}} := \max_{1 \leq i \leq m} \frac{\log p_i}{\log r_i}.$$
Principal assumptions and def. of $T(q)$

**Principal assumptions:**

(A1) $S$ satisfies SSP. That is

$$d := \min_{i \neq j} \text{dist} \{ S_i(\Lambda), S_j(\Lambda) \} > 0. \quad (21)$$

(A2) $p \neq (r_1^s, \ldots, r_m^s)$,

**Definition**

For a $q \in \mathbb{R}$, let $T(q)$ be the unique solution of the equation

$$\sum_{i=1}^{m} p_i^q r_i^{T(q)} = 1 \quad (22)$$

**Homework** Prove that $T''(q) > 0$ for all $q \in \mathbb{R}$. 
(a) The function $T(q)$

(b) The function $\alpha(q) := -T'(q)$

Figure: $m = 2$, $p = \left(\frac{3}{4}, \frac{1}{4}\right)$, $r = \left(\frac{1}{9}, \frac{1}{3}\right)$. 
The main Theorem

We assume that assumptions \((A1)\) and \((A2)\) hold for the IFS \(S\). Then the multifractal spectrum of the self-similar measure \(\nu = \Pi_* (p^\mathbb{N})\) is

\[
\mathcal{D}(\alpha) = \dim_H \{ x : d_\nu(x) = \alpha \} = \begin{cases} 
T^*(\alpha), & \text{if } \alpha \in [\alpha_{\text{min}}, \alpha_{\text{max}}]; \\
0, & \text{otherwise.}
\end{cases}
\]

where \(T^*\) is the Legendre transform of the convex function \(T\). That is

\[
T^*(\alpha) := \inf_q \left( T(q) + \alpha \cdot q \right). \tag{23}
\]
Example

Let us assume that \( m = 2 \) and \( \mathbf{p} = \left( \frac{3}{4}, \frac{1}{4} \right), \mathbf{r} = \left( \frac{1}{9}, \frac{1}{3} \right) \).
That is we consider the IFS

\[
\mathcal{S} = \left\{ \frac{1}{9} \cdot x, \frac{1}{3} \cdot x + \frac{2}{3} \right\},
\]

and we write \( \nu \) for the self similar measure with probabilities \( \mathbf{p} \). In this case we can find formulae for

See Figure ??.
Figure: Dimension spectrum $\mathcal{D}(\alpha)$ in the case when $m = 2$ and $p = \left(\frac{3}{4}, \frac{1}{4}\right)$, $r = \left(\frac{1}{9}, \frac{1}{3}\right)$. $\alpha_1 := \frac{\left(\sum_i p_i \log p_i\right)}{\left(\sum_i p_i \log r_i\right)} = \dimH \nu$, $\alpha_2 := \frac{\left(\sum_i r_i^s \log p_i\right)}{\left(\sum_i r_i^s \log r_i\right)}$
Definition

1. \( \mu_q := \left\{ p_1^q \cdot r_1^{T(q)}, \ldots, p_m^q \cdot r_m^{T(q)} \right\}^\mathbb{N} \), \( \nu_q := \prod_\ast (\mu_q) \).

2. **Definition of** \( \alpha(q) \):

\[
\alpha(q) := -T'(q) = \frac{\sum_{i=1}^{m} p_i^q r_i^{T(q)} \log p_i}{\sum_{i=1}^{m} p_i^q r_i^{T(q)} \log r_i}
\]

3. **Definition of** \( q(\alpha) \). For \( \alpha \in (\alpha_{\min}, \alpha_{\max}) \) we define the function \( q(\alpha) \) as the inverse function of \( \alpha(q) \). (\( T''(q) > 0 \) so this makes sense.)
Lemma

For \( \nu_q \) a.e. \( x = \Pi(i) \) the following two assertions hold

\[
d_\nu(x) = \lim_{n \to \infty} \frac{\log p_{i_1 \ldots i_n}}{\log r_{i_1 \ldots i_n}} = \alpha(q).
\] (24)

and

\[
d_\nu_q(x) = T(q) + q \cdot \alpha(q) \iff d_\nu(x) = \alpha(q).
\] (25)

The proof of the Lemma is a simple application of LLN and left as an exercise. Let

\[
f(\alpha) := T(q(\alpha)) + \alpha \cdot q(\alpha), E_\alpha := \{x \in \Lambda : d_{\nu_q(\alpha)}(x) = f(\alpha)\}.
\]
It follows from (25) that

\[ K_\alpha = \{ x : d_\nu(x) = \alpha \} = E_\alpha. \]

Using L.S. Young’s Theorem for the measure \( \nu_{q(\alpha)} \) we obtain that

\[ f(\alpha) = \dim_H(K_\alpha). \]

Now we prove that

\[ f(\alpha) = T^*(\alpha). \quad (26) \]

First observe that by differentiation a function

\[ q \rightarrow T(q) + \alpha \cdot q \]

attains its minimum at the \( q \), where \( \alpha = -T'(q) \), which is equal to \( \alpha(q) \) by definition.
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