SOME CHALLENGES IN THE THEORY OF (SEMI)-DISPERSING BILLIARDS

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ABSTRACT. Some challenging problems are explained arising in the theory of (semi)-dispersing billiards.
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1. INTRODUCTION.

The physically utmost interesting examples of hyperbolic billiards are *hard ball systems*. The big challenge: the Boltzmann-Sinai ergodic hypothesis claiming their *ergodicity* is almost that of the past. (As to the main steps and the state of affairs related to this celebrated hypothesis see section 2.)

With ergodicity being a *qualitative* property of dynamical systems, physicists are, in general, more interested in measurable, *quantitative* behavior, on the first place in *correlation decay*. Moreover, good correlation decay can lead to further important inference about the system, e. g. it can be helpful in deriving central limit theorem (CLT), or convergence to Brownian motion of some interesting physical processes, like the periodic finite horizon Lorentz process (cf. [BS 81], [BChS 91]) or its locally perturbed variants (cf. [DSzV 08b]). With the two-dimensional situation fairly well understood, the big problem of the theory is *correlation decay in higher-dimensions*. Bálint and Tóth have recently reached a breakthrough here, [BT 08]. Their result, however, hangs on a *plausible complexity condition*, whose verification I consider the top challenge of the theory. Much the more it is so since the general form of the local ergodicity theorem for dispersing billiards given by Bálint, Bachurin and Tóth, [BBT 08] also relies on the same condition. I am going to say more about this challenge in section 3.

In section 4 we introduce the dispersing *Penrose-billiard* or better *Penrose-Lorentz process* (in the plane with a finite horizon) and ask whether the diffusive behavior of the finite horizon planar Lorentz

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process with a periodic configuration of scatterers remains valid for the Penrose-Lorentz process. This question is connected to recent investigations of Lorentz processes, non-translationally-invariant wrt $\mathbb{Z}^2$. This section also contains an interlude: a question about the joint motion of two Lorentz discs.

For notions related to billiards the reader is advised to turn to the monograph of Chernov and Markarian [ChM 06] or to the collection of surveys [Sz 00].

2. THE BOLTZMANN-SINAI ERGODIC HYPOTHESIS

In 1962, Ya. G. Sinai in his lecture on the International Congress of Mathematicians in Stockholm (cf. [S 63]) formulated his form of Boltzmann's ergodic hypothesis:

Boltzmann-Sinai Ergodic Hypothesis. *any number $N \geq 2$ of — sufficiently small — elastic hard balls moving on the $\nu = 2$ or $\nu = 3$-dimensional torus is ergodic on the submanifold of the phase space determined by the trivial invariants of the motion*.

(Since the general multidimensional case $\nu \geq 2$ occured to be the same hard as the three-dimensional one, we can also call the general conjecture the Boltzmann-Sinai ergodic hypothesis.) Sinai — in his classical work, [S 70] — settled it for $N = \nu = 2$ and later he and Chernov, [SCh 87] for $N = 2$. It is easy to see that hard ball systems are, in general, isomorphic to semi-dispersing billiards (and if $N = 2$, even to dispersing ones). At this point it is worth mentioning that the proof — by Krámli, Simányi and the present author — of ergodicity of the first semi-dispersing but not strictly dispersing billiard appeared in the *second volume of Nonlinearity* precisely twenty years ago, see [KSSz 89]. The novel ideas of that work has led to a bunch of more and more sophisticated works and as a result of them the story of the Boltzmann-Sinai ergodic hypothesis has been conditionally completed by Simányi, [Sim 08] twenty years after the appearance of the aforementioned Nonlinearity paper. In fact, he has to assume the fulfilment of a condition of the local ergodicity theorem for semi-dispersing billiards: that of the *Chernov-Sinai ansatz* (cf. [SCh 87] and [KSSz 90]; as to more details see [Sz 08]). Of course, apart from hard ball systems on tori there is an abundance of (semi)-dispersing billiards whose ergodicity is a challenging open problem. Here I only mention two of them closest to the Boltzmann-Sinai ergodic hypothesis.

Problem 1. *Hard balls in a box.* The problem seems to be harder than the Boltzmann-Sinai hypothesis. One reason is that the torus has more symmetries, in particular it has the group structure that was heavily exploited in the algebraic methods. Another, more technical reason is that by having less invariants of motion (total momentum is not conserved any more) hyperbolicity of an orbit itself is already a more stringent property.

Problem 2. *Erdőtarcsa conjecture*, [SSz 00]. It is a general conjecture for the ergodicity of cylindrical billiards given in terms of the transitivity of a Lie group defined by the cylindric scatterers. Since the class of cylindric billliards contains hard ball systems, the Erdőtarcsa conjecture of Simányi and Szász is stronger than the Boltzmann-Sinai hypothesis. (Cylindric billiards were introduced in [Sz 93], where a naïve form of the Erdőtarcsa conjecture had already been formulated.)

3. THE COMPLEXITY CONDITION FOR DISPERSING BILLIARDS

To estimate correlation decay rate for axiom A systems *Markov partitions* have been most effective since they were in a sense easy to be constructed and further there was an evident gap in the spectrum of the transfer operator. For singular systems, like billiards, however, Markov partitions are necessarily countable and their construction — as it was originally designed by Bunimovich and Sinai, [BS 81] — was fairly complicated and rigid already in the two-dimensional case. Still restricted to $d = 2$, Bunimovich, Chernov and Sinai, [BChS 91] substituted Markov partitions by more flexible objects, like *Markov sieves*: these are sequences of finite families of subsets and their construction has been simpler and in a sense more robust. However, they still have only led to *streched exponential decay* of correlations. A breakthrough was obtained in 1998 by the *Markov-tower* construction of Young, [Y 98] that provided *spectral gap and exponential decay*, at least in the planar case.

After the appearance of Young's tower construction alternative methods have been found for obtaining stronger and finer stochastic properties. From among them the Banach-space/transfer operator methods (by Baladi, Gouëzel, Keller, Liverani, Tsujii, ...) or the geometric ones (by Chernov, Dolgopyat and Young) based on growth lemmas and coupling lemmas.

A natural goal was then to *extend Young's construction to the multidimensional case*. The delicate work of Bálint and Tóth, [BT 08] has unfolded the situation: they have, indeed, constructed Young towers for multidimensional dispersing billiards with a finite horizon. Their result is, however, conditional since they need an additional geometric assumption related to the complexity condition to be explained later. This condition is plausible for at least generic billiard tables, but so far they could not establish it for any (!) model. The solution of this problem is a 'must' of the theory (and not only for proving correlation decay!).

Let us turn to the formulation of the complexity condition. For simplicity let us consider a (strictly) dispersing billiard on the d-torus having finite horizon and no corner points. Denote it by $(M = \partial\tilde{Q} \times S_+, T, \mu)$ where $Q = \mathbb{T}^d \setminus \cup_{j=1}^p O_j$ is the billiard table with $O_j, 1 \leq j \leq p$, the disjoint, C^3-smooth, strictly convex scatterers, S_+ the hemisphere of outgoing unit velocities, T the billiard map, and μ the invariant Liouville-measure. (By denoting $x = (q, v)$ and $Tx = (x^+, v^+)$ the billiard map T is defined via $q^+ = \inf\{t > 0 |\ q + tv \in \partial Q\}$ and $v^+ = v - 2 < v, n^+ > v$ where v^+ is the outer normal to ∂Q in the point q^+.) Sinai, in his classical work [S 70], worked out the fundamental ideas and tools of hyperbolic theory for billiards, a key example of systems with singularities. He there formulated the basic philosophy as to the possibility of such a theory by saying *'expansion should prevail partitioning'*. Another expression by Sinai of the same idea was the following: for (an unstable invariant curve) to get partitioned it should first get expanded. For planar billiards these curves are one-dimensional and the geometry of their expansion and partitioning is simple (essentially they only have length but no shape!).

Denote by $S_0 \subset M$ the set of all tangential collisions, and take the union $S_0^n = \cup_{j=0}^n T^j S$ of its images under the first n iterates of T. For $n \geq 1$ consider all closures of the smooth components $S_1^{(n)}, S_2^{(n)}, \ldots, S_{l_n}^{(n)}$ of S_0^n. These submanifolds partition the phase space into connected components. Further for every phase point $x \in \cup_{l=1}^{l_n} S_l^{(n)}$ it makes sense to define the number $K_n(x)$ of connected components of M in sufficiently small neighborhoods of x. Denote $K_n = \sup_{x \in \cup_{l=1}^{l_n} S_l^{(n)}} K_n(x)$.

Definition 1. *(***Complexity condition,** [Y 98]*). Let $\Lambda > 1$ be the uniform minimal expansion of the billiard map over unstable vectors. The complexity condition requires the existence of a $\lambda < \Lambda$ and an n_0 such that $\forall n \geq n_0\ \ K_n < \lambda^n$.*

Then the theorem of Bálint and Tóth says

Theorem 1. (Bálint-Tóth, [BT 08]**)** *If a finite horizon dispersing billiard without corner points satisfies the complexity condition, then for arbitrary* *pair of Hölder functions f, g of the phase space there exist constants $C, a > 0$ such that $\forall n \geq 1$*

$$|\int_M f(x)g(T^n x)d\mu - \int_M f(x)d\mu \int_M g(x)d\mu| \leq Ce^{-an}$$

As formulated above the complexity condition seems to be pretty technical. Before substituting it with a more transparent form, let us return briefly to Sinai's philosophy. Its beautiful quantitative phrasing was given by Chernov, [Ch 99] in the form of *growth lemmas*. The growth lemmas grasp the multidimensional shape of the smooth pieces of images of invariant manifolds in an efficient way. The complexity condition is actually used to establish them. At present I think that requiring and using it is inevitable. One more remark: as observed by [BBT 08], the growth lemmas, and consequently the complexity condition, too, do also play a fundamental role in proving the local ergodicity theorem of Chernov and Sinai for general multidimensional dispersing billiards (cf. [SCh 87], [KSSz 90] and [B 05]). Indeed, so far this theorem has only been proved under the additional assumption: *the boundaries of the scatterers are algebraic* (see [BChSzT 02]), though everyone expects that this is only a technical condition, and the statement holds in the C^3-category.

For obtaining the promised transparent form of the complexity condition assume that the scatterers $O_j \in \mathbb{T}^d : 1 \leq j \leq p$ are strictly convex and algebraic, for instance spheres. Instead of the torus it might be simpler to think of the case of $O_j \in \mathbb{R}^d : 1 \leq j \leq p$ and one can make a simple heuristic calculation that can be made rigorous easily. Note that the dimension of the phase space of the billiard flow is $2d - 1$. By introducing the vectors $c_j \in \mathbb{R}^d : 1 \leq j \leq p$ and considering the shifted scatterers $O_j + c_j \in \mathbb{R}^d : 1 \leq j \leq p$, one expects that for typical configurations of the centers $c_j : 1 \leq j \leq p$ no billiard orbit exists which would be tangent at least $2d$ times to the scatterers. This can be proven rigorously locally in the neighborhood of an orbit which does not have recollisions with any of the scatterers. The transparent for of the complexity condition requires the same when the scatterers are on $\mathbb{T}^d$. The difficulty is, of course, that then a common tangent orbit can be tangent to the same scatterer several times!

Transparent form of the complexity condition. Consider a toric billiard as above. There exists a constant $C(d)$, depending only on d, such that for typical configuration of the centers no billiard orbit exists with more than $C(d)$ tangential collisions. We note that the complexity condition would follow if the constant $C(d)$ depended on the billiard table, too.

The authors of [BT 08] plan to publish their arguments that, indeed, this property implies the complexity condition and, moreover, an example of a multidimensional dispersing billiard where the complexity condition is not fulfilled (personal communication).

4. THE PENROSE BILLIARD OR THE PENROSE-LORENTZ PROCESS

The starting point of our last problem is the CLT for the Lorentz process.

Theorem 2. (Bunimovich-Sinai, [BS 81]**)**: *The diffusively scaled variant $W_N(t)$ of the periodic, finite-horizon, planar Lorentz process converges weakly to a Wiener process $W_{\mathcal{D}^2}(t)$ with a non-degenerate covariance matrix $\mathcal{D}^2$.*

Immediately after the appearence of this result, Sinai formulated the following conjecture:
Sinai's conjecture, 1981: *The same statement holds for the locally perturbed periodic Lorentz process (finite horizon, $d = 2$)*

Here local perturbation means that that the periodic scatterer configuration is modified in a bounded domain in a rather arbitrary way. One, should, of course, conserve the dispersive feature of the — noncompact — billiard table, the local perturbation is otherwise sufficiently arbitrary. Recently an affirmative answer has been obtained to the conjecture, namely we have

Theorem 3. (Dolgopyat, Szász, Varjú, [DSzV 08b]**)** *For the locally perturbed finite-horizon, planar Lorentz Process, as $N \to \infty$, $W_N(t) \Rightarrow W_{\mathcal{D}^2}(t)$ (weak convergence in $C[0,\infty]$), where $W_{\mathcal{D}^2}(t)$ is the Wiener process with the non-degenerate covariance matrix $\mathcal{D}^2$. The limiting covariance matrix coincides with that for the unmodified periodic Lorentz process.*

The authors of the aforementioned work also discuss other — in a sense local — alterations of the model as compared to periodicity. The next problem I formulate is a different kind of digression from periodicity in the direction of quasi-periodicity.

Interlude: a related problem: two Lorentz discs. Before going over to the problem on the Penrose-lattice, let me formulate a problem partially related to both the local perturbed Lorentz process and to the multidimensional case. An implication of the CLT for Hölder functions of a finite horizon, planar, dispersing — in other words Sinai- — billiards, is the weak convergence of the diffusively scaled periodic Lorentz process to the Brownian motion process. This result can also be interpreted as follows: consider a small disc particle moving among the periodic scatterers of the previous Lorentz process (this dynamics will be called a *Lorentz disc*). Then the diffusively scaled trajectory of the disc converges to a Brownian motion as well (it is isomorphic to a Lorentz process with larger scatterers). For various particle dynamics the joint rescaled motion of two particles was shown to asymptotically behave as independent (see for instance [Sz 80]). For two Lorentz discs, however, a different picture is expected. The reason is twofold: on the one hand, at collisions of the two discs, there occurs an exchange of the energies between the particles and, on the other hand, these collision are extremely scarce. These effects cause the limiting processes of the individual particles to be mixtures of Brownian motions. This picture have been shown to be precise for toy models of the Lorentz discs: random walks with internal states (work in preparation). When extending them to physical models it is reasonable to expect that the energy transfer formula (i. e. Equ (5)) of [GG 08] will describe the change of energies in the rare moments when the two particles actually collide.

Let us now go to our goal, to the definition of the Penrose-Lorentz process. Consider a Penrose lattice in the plane. As described e. g. in [TP 08], a Penrose tiling can be constructed by using two types of isosceles triangles $\{L, S\}$. By denoting $\theta = \frac{\pi}{5}$ and $\tau = \frac{1+\sqrt{5}}{2}$ the angles of L are $\theta, 2\theta, 2\theta$ and its sides are $1, \tau, \tau$, whereas the angles of S are $\theta, \theta, 3\theta$ and its sides are $1, 1, \tau$. The symbolic rules of the construction force these triangles occur in pairs, only, a triangle reflected wrt to a τ-side (L) or a 1-side (S) respectively. The arising quadrangles are called kites and darts, respectively. It is known that by fixing a vertex of the Penrose lattice, only 7 types of local neighborhoods of any vertex are possible.

I call a finite horizon dispersing billiard in the plane a **Penrose-Lorentz process** if its scatterer configuration possesses the translational invariance of the Penrose lattice. By this I mean that in each tile of type L (and of type S, too) the scatterer configuration is isomorphic. The construction is easy: around each vertex one fixes a circle of radius $r < \frac{1}{2}$, centered at the vertex. The fact that this billiard has finite horizon follows from the aperiodicity properties of the Penrose lattice. Let us denote the corresponding dynamical system by $(M_P = \partial\tilde{Q}_P \times S_+, T_P, \mu_P)$ where $Q_P = \mathbb{R}^2 \setminus \cup_{j=1}^{\infty} O_j$ is the billiard table with $O_j, 1 \leq j \leq \infty$, the disjoint circular scatterers, T_P the billiard map, and μ_P the invariant (infinite) Liouville-measure.

Denote by $S_n(x)$ the position of the Lorentz particle in the Penrose-Lorentz process in the moment of $n'th$ reflection.

Conjecture. *By selecting the initial phase point of the Penrose-Lorentz process according to a probability measure absolutely continuous wrt to the Liouville measure the diffusively scaled variant $W_N(t)$ of the Penrose-Lorentz trajectory converges weakly to a non-degenerate, rotation-invariant Wiener process.*

The conjecture is supported by the common understanding that the results of random walk theory provide a good guess what can be expected for the Lorentz process (and often its methods are also most useful). (As to recent examples cf. [DSzV 08a], [DSzV 08b], [P 08a], [P 08b]). On the basis of this belief I remark that for nearest neighbor random walks on the Penrose lattice the convergence to the Wiener process has been proven. Indeed, as early as in 2000, M. Kunz, [K 00] already established a conditional result: under the condition that harmonic coordinates exist, $\frac{S_n}{\sqrt{n}}$ is asymptotically normal with zero mean and a rotation invariant covariance matrix. What is more, as A. Telcs has informed me, there are general results for random walks on graphs (in particular, those by Delmotte, [D 99] and by Hambly-Kumagai, [HK 04]), which — combined with a recent observation of Solomon, [So 07] claiming that the Penrose lattice is biLipschitz to the integer lattice — provide the asymptotic normality unconditionally for a random walk on the Penrose lattice.

As to a possible proof I note that recent strong methods: growth lemmas, generalizations of Young's coupling, etc. of Chernov and Dolgopyat, [ChD 07] should work more generally as stated for purposes of concrete applications for billiards (op. cit) or periodic Lorentz processes ([DSzV 08a]). For instance, I expect that the growth lemmas are also valid if the lengths of free jumps, and the curvatures of the scatterers are bounded both from below and from above.

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