Abstract In [**?**], D. Szász and A. Telcs have shown that for the diffusively scaled, simple symmetric random walk, weak convergence to the Brownian motion holds even in the case of local impurities if $d \geq 2$. The extension of their result to finite range random walks is straightforward. Here, however, we are interested in the situation when the random walk has unbounded range. Concretely we generalize the statement of [**?**] to unbounded random walks whose jump distribution belongs to the domain of attraction of the normal law. We do this first: for diffusively scaled random walks on $\mathbf{Z}^d$ ($d \geq 2$) having finite variance; and second: for random walks with distribution belonging to the non-normal domain of attraction of the normal law. This result can be applied to random walks with tail behavior analogous to that of the infinite horizon Lorentz-process; these, in particular, have infinite variance, and convergence to Brownian motion holds with the superdiffusive $\sqrt{n \log n}$ scaling.

Locally Perturbed Random Walks with Unbounded Jumps

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1 Introduction

Our goal in this paper is to show that local impurities do not influence the - appropriately scaled - weak limit behavior of random walks on $\mathbf{Z}^d$ ($d \geq 2$). In [?], D. Szász and A. Telcs have shown that for the diffusively scaled, simple symmetric random walk, weak convergence to the Brownian motion holds even in the case of local impurities if $d \geq 2$. The extension of their result to finite range random walks is straightforward. Here, however, we are interested in the situation when the random walk has unbounded range (for simplicity, we always assume that the distribution of the jumps is centered, i.e. their expected value is 0). Concretely we generalize the statement of [?] to unbounded random walks whose jump distribution belongs to the domain of attraction of the normal law. We do this first: for diffusively scaled random walks on $\mathbf{Z}^d$ $d \geq 2$ having finite variance, and second: for random walks with distribution belonging to the non-normal domain of attraction of the normal law (we note that, if $d = 1$, this domain consists of distributions which have infinite variance but $L(x) = \int_{|u| \leq x} u^2 dF(u) \to \infty$ is a slowly varying function; in this case partial sums converge to the gaussian law by using the slightly superdiffusive scaling B_n, where B_n is determined by the relation $\frac{nL(B_n)}{B_n^2} \to 1$. For the reader's convenience, a summary of related definitions and results can be found in the Appendix). The idea of [?] served as an intuitive background for proving that the diffusively scaled, locally perturbed, planar, finite horizon Lorentz process converges weakly to the Brownian motion (cf. [?] and [?]). Our second result is hoped to provide a starting point to prove the corresponding statement for the infinite horizon Lorentz process, recently a widely studied object (cf. [?], [?], [?], [?]). (In fact, for stochastic models of the infinite horizon Lorentz processes other types of perturbations should also be taken into account, an object of future research.)

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In Section 2, we are going to state our general theorems, Section 3 contains some lemmas and definitions, and Section 4 proves the theorems using these. The proofs of the lemmas can be found in Section 5. Finally, Section 6 is devoted to comments.

2 Main results (in particular, the generalization of [?] to the case of infinite horizon)

The main difficulty in generalizing Theorem 1 of [?] to the case of random walks with unbounded jumps is that in this case the coupling argument of [?] breaks down. Though the structure of our proofs is similar to that used in [?], for avoiding the aforementioned difficulty one needs novel ideas (as seen, in particular, in the proofs of our lemmas in Section 5).

For avoiding unnecessary complications we suppose throughout the whole paper: (i) the dimension $d \geq 2$; (ii) the aperiodicity of the random walk (i. e. if we denote by $P_n(z \to x)$ the n-step transition probability of the random walk, then we assume: if there is an m such that $P_m(z \to x) > 0$, then we can choose n_0 such as for all $n > n_0$, $P_n(z \to x) > 0$). Finally $||.||$ denotes the maximum (L_∞) norm for vectors.

Definition 1 Let $y_1, y_2, \ldots$ be independent, identically distributed (or briefly iid) random variables with

$$P(y_i = y) = \underline{P}(y) \qquad y \in \mathbb{Z}^d.$$

The stochastic process $Y_0, Y_1, \ldots$ defined by

$$Y_n = y_0 + \sum_{i=1}^{n} y_i$$

where $y_0 = z \in \mathbb{Z}^d$, is a *random walk* (or briefly RW). The measure defined by this random walk will be denoted by $\underline{P}_z$, and the transition probabilities of the process will be denoted by $\underline{P}(x, y)$ ($\underline{P}(x, y) = \underline{P}(0, y - x) \stackrel{def}{=} \underline{P}(y - x)$).

As usual, the random walk orbit determines a continuous time stochastic process $\eta_n^Y : [0, 1] \to \mathbb{R}$ with continuous trajectories in the following way: $\eta_n^Y(t) = n^{-1/2}Y_{nt}$ if $t = 0, \frac{1}{n}, \frac{2}{n}, \ldots, 1, \ldots$, and it is linear between these points. It is well known that if $\underline{P}$ is such that second moments of the jumps are finite, i.e.

$$\sum_{x \in \mathbb{Z}^d} \underline{P}(x)||x||^2 < +\infty,$$

then, as $n \to \infty$, η_n^Y converges weakly to $W_{\sum}(t)$, a d-dimensional Wiener-process in the space $C_d[0, \infty)$ (with $d \times d$ covariance matrix $\sum$ equal to that of the random vector y_i). Weak convergence in $C_d[0, \infty)$ will be denoted by $\Rightarrow$. The proof of this result (d-dimensional Donsker functional central limit theorem) can be found in [?] page 106, Theorem 4.3.5.

Remark 1 Since our arguments that show weak convergence in $C_d[0, 1]$ also imply that in $C_d[0, \infty)$ in a standard way (cf. [?]), we often will only formulate our statements for $C_d[0, 1]$.

Definition 2 Let P be a transition probability matrix on $\mathbb{Z}^d$ (so $P(x, y)$ is not necessarily equal to $P(0, y - x)$). We call the time-homogeneous Markov process X_i, $i = 0, 1 \ldots$ with transition probabilities P a *random walk in an inhomogeneous medium.*

Definition 3 If X_n is a random walk in an inhomogeneous medium and there exists a finite set $A \subset \mathbb{Z}^d$ such that for all $u \notin A$, $v \in \mathbb{Z}^d$, $P(u, v) = \underline{P}(u, v)$, then we call X_n a *random walk with local impurities* (or briefly RWwLI).

Let us define a directed graph $G = (\mathbb{Z}^d, E)$, where $(E = \{(u, v) | P(u, v) \neq 0\})$. In both our theorems we assume:

Assumptions:

(i) let X_n be a RWwLI , where the starting point z lies in the infinite, strongly connected component Q of G (a directed graph Q is strongly connected if there is a directed path from each vertex to each other vertex in Q, a strongly connected component of G is a maximal strongly connected subgraph); for simplicity, assume also that $0 \in Q$;

(ii) assume that there is an $\epsilon > 0$ that for all the impurities, the jump from the impurity has a distribution whose ϵth moment exists (i.e. $E(||J||^\epsilon) < +\infty$, where J is the jump from an impurity).

Theorem 1 *Define $\eta_n^X : [0, 1] \to \mathbb{R}$ analogously to $\eta_n^Y(t)$. Suppose that the second moment of the jumps (with respect to $\underline{P}$) exists, their expected value is 0 and the RW is aperiodic. Then, as $n \to \infty$,*

$$\eta_n^X(t) \Rightarrow W_{\sum}(t), \qquad t \in [0, 1].$$

Definition 4 We call a random variable (or random vector) X with $\mathbb{E}||X||^2 = \infty$ B-type if its expected value is 0, and it belongs to the domain of attraction of the normal law. (As a consequence, for partial sums of iid B-type summands the scaling is larger than $\sqrt{n}$, i.e. $\lim\limits_{n \to \infty} \frac{B_n}{\sqrt{n}} = \infty$). The reader is reminded that some facts from classical limit theory of probability, e. g. the definition of domain of attraction, are collected in the Appendix.

Theorem 2 *Suppose that the distribution of the RW jumps is B-type with scaling B_n, their expected value is 0 and the RW is aperiodic. Define $\eta_n^{X'} : [0, 1] \to \mathbb{R}$ as*

$$\eta_n^{X'}(t) = B_n^{-1} X_{nt} \text{ if } t = 0, \frac{1}{n}, \frac{2}{n}, \ldots, 1,$$

and let it be linear between these points. Then as $n \to \infty$,

$$\eta_n^{X'}(t) \Rightarrow W_{\sum}(t), \qquad t \in [0, 1].$$

Remark 2 Of course, it may also occur that partial sums of iid random vectors would 'naturally' scale differently in different directions. This is, for instance, the case in Theorem 7 of [?]: they consider a two-dimensional periodic Lorentz process where all the collision-free trajectories are parallel to the x-axis and the scaling is $\sqrt{n \log n}$ in the direction of the x-axis whereas it is $\sqrt{n}$ in the direction of the y-axis. The reader can convince him/herself that general case is analogous: in some directions one has to use the strongest scaling and then in some orthogonal directions the next to the strongest ones, etc. and our methods are also applicable in this situation.

Remark 3 For $d = 1$, Theorem 1 is not true; cf. [**?**].

Remark 4 In Theorem **??**, $\sum$ is the same as the covariance matrix of the jumps of the RW (having distribution $\underline{P}$). For Theorem 2, $\sum$ is still determined by the distribution of the RW jumps (see Theorem 4.2 of [**?**] in the Appendix).

3 Preliminary notes to the proofs of Theorems ?? and ??

Before the rigorous discussion we want to show the idea of the result. If $d \geq 3$, then for the simple symmetric random walk, Pólya's theorem says that with a probability 1, the number of returns into the origin — or into a finite set A — is finite. This can be simply proven for any non-degenerate random walk. For instance, for $d = 3$, once the Local Central Limit Theorem ([**?**], page 25) holds, stating that the probability of return into the origin in the nth step is $p_n = O(n^{-3/2})$, then the series p_i is summable. Thus the expected number of returns is finite. Therefore, in case of a finite modification, the random walk leaves the set A after a finite time. Consequently, in the limit, the effect of the modification vanishes.

Much more interesting is the case $d = 2$, when the random walks we are interested in are recurrent. Again, once for some sequence $\{B_n > 0\}_{n \geq 1}$ $\frac{Y_n}{B_n}$ has a limit law, then under some additional conditions, its local version also holds. It implies that the expected number of returns into a finite set until time n is $O(\sum_{j=0}^{n} \frac{1}{B_j^2}) = O(\sum_{j=0}^{n} \frac{1}{j})$ which is always $O(\log n) = o(\sqrt{n})$. The expected time spent in A during one visit is uniformly bounded and since the normalizing factor is at least of order $\sqrt{n}$, the previous conclusion is also true.

Now, we are going to start with definitions and our key lemma (its proof can be found in section 5).

Definition 5 The impurities are local, so we can select and fix an $N \in \mathbb{N}$ such that all the impurities lie in the cube

$$K_N = \left[-N - \frac{1}{2}; N + \frac{1}{2}\right]^d \subset \mathbb{R}^d \quad \text{i.e., } A \subset K_N.$$

Definition 6

$$\rho_n = \sum_{i=0}^{n} \chi_{X_i \in K_N}$$

i.e. the time spent by the RWwLI in K_N until time n.

Lemma 1 *If $d \geq 2$, and $z \in Q$, then, as $n \to \infty$,*

$$E_z(\rho_n) = O(\log n).$$

4 Proofs of Theorems ?? and ?? assuming Lemma ??

Proof of Theorem ?? We are going to write the proof for dimension $d \geq 2$ (even though the $d \geq 3$ case could be proven simply, as indicated before). With the help of X_n we define a new process Z_n. Let $Z_0 = z$ and $Z_{n+1} - Z_n = X_{n+1} - X_n$ if $X_n \notin K_N$, while if $X_n \in K_N$, then let $Z_{n+1} - Z_n$ be independent of $X_0, X_1, \ldots X_n$

and of $Z_0, Z_1, \ldots Z_n$, and let $P(Z_n, Z_{n+1}) = \underline{P}(Z_n, Z_{n+1})$. Then it is clear that Z_n is a random walk with transition matrix $\underline{P}$. Let's define $\eta_n^Z(t)$ analogously as $\eta_n^Y(t)$ in Section 2, then

$$\eta_n^Z(t) \Rightarrow W_{\sum}(t)$$

Thus in order to establish that

$$\eta_n^X(t) \Rightarrow W_{\sum}(t)$$

also holds, because of the piecewise linearity of $\eta_n^X(t)$ and $\eta_n^Z(t)$, it is sufficient to show that

$$n^{-1/2} \cdot \sup_{t \in [0,1]} ||X_{[nt]} - Z_{[nt]}|| \to 0 \text{ as } n \to \infty.$$

Observe that

$$n^{-1/2} \cdot \sup_{t \in [0,1]} ||X_{[nt]} - Z_{[nt]}|| \leq n^{-1/2} \cdot \sum_{i=1}^n ||(X_i - X_{i-1}) - (Z_i - Z_{i-1})|| =$$

$$= n^{-1/2} \cdot \sum_{i: X_{i-1} \in K_N} ||(X_i - X_{i-1}) - (Z_i - Z_{i-1})||$$

Let us define a sequence of random variables $J_k = ||k\text{th step from } K_N||$, $k \in \mathbb{Z}^+$. Then our task is to show that

$$n^{-1/2} \cdot \sum_{k=1}^{\rho_n} J_k \Rightarrow 0$$

Firstly, we will show this for the case when K_N consists of only one point, and then we show it for the general case.

In the one - point case, J_k are independent and they have the same distribution, which, by assumption, has $E(J_k^\epsilon) < \infty$ for some $\epsilon > 0$, by a simple argument we can also suppose that $\epsilon < 1$. Let us call $E(J_k^\epsilon) = K$, then by Markov - inequality, we have $P(J_k^\epsilon > h) < K/h$ $(h > 0)$, therefore $P(J_k > h) < K/h^\epsilon$. Let us define a new random variable J' as $P(J' > h) = 1$ if $h \leq 0$ and $P(J' > h) = \min(1, K/h^\epsilon)$ if $h > 0$. Then we define the sequence J_k' with the same distribution as J', and can see that for every $\gamma \geq 0$,

$$P\left(n^{-1/2} \cdot \sum_{k=1}^{\rho_n} J_k > \gamma\right) \leq P\left(n^{-1/2} \cdot \sum_{k=1}^{\rho_n} J_k' > \gamma\right)$$

therefore it is sufficient to show that the latter converges to zero as $n \to \infty$. Then we can easily verify that this distribution of J' belongs to the domain of attraction of stable law with parameter $\alpha = \epsilon$. Therefore if we sum k random variables of this distribution, and note the sum by S_k, then $\frac{S_k}{k^{1/\epsilon}} \Rightarrow S(\epsilon, \beta)$, where $S(\epsilon, \beta)$ is a stable law.

Now we can write

$$P\left(n^{-1/2} \cdot \sum_{k=1}^{\rho_n} J_k' > \gamma\right) \leq P\left(n^{-1/2} \cdot \sum_{k=1}^{\rho_n} J_k' > \gamma \,\middle|\, \rho_n > \log^2(n)\right) \cdot P\left(\rho_n > \log^2(n)\right) +$$

$$+ P\left(n^{-1/2} \cdot \sum_{k=1}^{[\log^2 n]} J_k' > \gamma\right)$$

By Lemma **??**, $E(\rho_n) = O(\log n) = o(\log^2 n)$. Hence the first term tends to zero by the Markov - inequality. Therefore we only need to show that the second term tends to zero too.

$$P\left(n^{-1/2} \cdot \sum_{k=1}^{[\log^2 n]} J'_k > \gamma\right) = P\left(\left(\log^{-2/\epsilon}(n) \cdot \sum_{k=1}^{[\log^2 n]} J_k\right) \cdot \frac{n^{-1/2}}{\log^{-2/\epsilon}(n)} > \gamma\right)$$

We have $P\left(\log^{-2/\epsilon}(n) \cdot \sum_{k=1}^{[\log^2 n]} J_k > x\right) \rightarrow P\left(S(\epsilon, \beta) > x\right)$ for every $x > 0$ as $n \rightarrow \infty$, so it is obvious that $P\left(n^{-1/2} \cdot \sum_{k=1}^{[\log^2 n]} J'_k > \gamma\right) \rightarrow 0$ as $n \rightarrow \infty$.

The second case is when K_N consists of several points. First we denote by V_n the set $x_1, x_2, \ldots x_n$ of the first n points in K_N hit by X_j. Let W_n be the set of all possible V_n that occur with probability greater that zero. Then we can write

$$P\left(n^{-1/2} \cdot \sum_{k=1}^{\rho_n} J_k > \gamma\right) = \sum_{V_n \in W_n} P\left(n^{-1/2} \cdot \sum_{k=1}^{\rho_n} J_k > \gamma \,\middle|\, V_n\right) P(V_n)$$

Now it is sufficient for us to show that $P\left(n^{-1/2} \cdot \sum_{k=1}^{\rho_n} J_k > \gamma \,\middle|\, V_n\right) \rightarrow 0$ as $n \rightarrow \infty$ for every $V_n \in W_n$.

If we fix V_n, then J_k are conditionally independent. Moreover, we can see that $J_k|V_n = J_k|\{x_k, x_{k+1}\}$, so it only really depends on the kth and $k + 1$th step in K_N. For fixed x_k, $J_k|x_k$ has finite ϵth moment for some $\epsilon > 0$ (independent of k and x_k). If

$$K = \max_{x \in K_N \cap Q} E(J_k^\epsilon | x_k = x),$$

then we see that $P\left(J_k > h | x_k\right) < K/h^\epsilon$. But

$$P(J_k > h | x_k) = \sum_{x \in K_N} P(x_{k+1} = x | x_k) \cdot P(J_k > h | x_k, x_{k+1}).$$

$P(x_{k+1} = x | x_k)$ is independent of k, it only depends on the two points in K_N, so it has at most $|K_N|^2$ different values. We only need to consider cases when it is not zero, because if it is zero, than it can never occur in any $V_n \in W_n$. Therefore there is a positive $p > 0$ such that $P(x_{k+1} = x | x_k) > p$ if it is not zero. So

$$P(J_k > h | V_n) < \frac{K}{ph^\epsilon}$$

From here, because of the conditional independence of J_k, we can follow the proof of the one - point case.

The condition of existence of ϵth moment seems difficult to avoid, since if we take the case of a distribution for J such that the cumulative distribution function has the form $F(x) = 1 - \frac{1}{\log^2(e+x)}$ for $x > 0$ and 0 for $x \leq 0$, then we can observe the sum of $\log n$ independent random variables of this kind, divided by $\sqrt{n}$, will not converge to zero. Nevertheless, it is possible that in this case the random walk returns less frequently than $O(\log n)$ times in n steps, but this seems difficult to prove.

Proof of Theorem ?? The proof is very similar to that of Theorem **??**, this time we define Z_n the same way as we have done in the previous proof, only now we define $\eta_n^Z(t) = B_n^{-1} Z_{nt}$ if $t = 0, \frac{1}{n}, \dots 1$, and linear between these points, then

$$\eta_n^Z(t) \Rightarrow W(t) \text{ for } t \in [0, 1].$$

From Lemma **??** we know that $E_z(\rho_n) = O(\log n)$, and using similar arguments as in the previous proof we can show that $B_n^{-1} \sup_{0 \le t \le 1} ||X_{[nt]} - Z_{[nt]}||$ tends to zero in probability as $n \to \infty$. This proves the statement of our theorem.

5 Proofs of lemmas

The proofs of Theorem 1 and 2 are based on Lemma **??**. In order to prove this, we need a few new definitions and lemmas.

Definition 7

$$\theta_n = \sum_{i=0}^{n} \delta(Y_i, 0)$$

(where δ is the Kronecker symbol), i.e., θ_n is the number of visits to the origin in the first n steps of the simple symmetric random walk.

Definition 8 For all $z \in K_N \cap Q$, consider a RWwLI such that $X_0 = z$. Denote

$$\tau_z = \min\{k \in \mathbb{N} | X_k \notin K_N\}$$

i.e. τ_z is the first exit time of the RWwLI from the set K_N.

Definition 9 Denote by ν_n the number of pure 1 blocks in the sequence

$$\chi_{X_0 \in K_N}, \chi_{X_1 \in K_N}, \dots, \chi_{X_n \in K_N}$$

i.e. ν_n is the number of entrances into K_N until time n.

We also define the slightly different random variable $\overline{\nu}_n$: it is the number of pure 1 blocks in the sequence

$$\chi_{x_0 \in K_N}, \chi_{x_1 \in K_N}, \dots, \chi_{x_m \in K_N} \tag{1}$$

where m is chosen so that the number of 0s in the sequence (**??**) equals n; i. e. $\overline{\nu}_n$ is the number of entrances into K_N until the first n steps outside K_N.

Definition 10 Denote by $B \subset Q$ a finite simply connected set, i.e., for all $x, y \in Q \setminus B$ there is a directed path from x to y in the subgraph on $Q \setminus B$. Let

$$S_B = \min\{k \in \mathbb{N} | X_k \in B\}$$

and

$$T_B = \min\{k \in \mathbb{N} | Y_k \in B\}.$$

These are called hitting times of B by the processes X_n and Y_n, respectively.

Lemma 2 *Let $H \subset (Q \setminus K_N)$ be a finite set. Then there exists a constant $C_{H,K_N} > 0$ (independent of z and n), such that for all $z \in H$,*

$$\frac{\underline{P}_z(T_{K_N} > n)}{\underline{P}_0(T_{\{0\}} > n)} > C_{H,K_N}$$

We shall denote by E_z (and $\underline{E}_z$) expectations with respect to P_z (and $\underline{P}_z$).

Lemma 3 *If $d \geq 2$, then, as $n \to \infty$,*

$$\underline{E}_z(\theta_n) = o(\sqrt{n})$$

Lemma 4 *If $d \geq 2$, then there is a $H \subset (Q \setminus K_N)$ finite set, and a $D_{H,K_N} > 0$ constant (independent of z and n), such that for every $z \in Q$,*

$$E_z(\nu_n) \leq E_z(\overline{\nu}_n) \leq \frac{D_{H,K_N}}{\min\limits_{b \in H} \underline{P}_b(T_{K_N} > n)}$$

Corollary 1 *If $d \geq 2$, then, as $n \to \infty$, $E_z(\overline{\nu}_n) = O(\log n)$.*

Now we can start the proofs.

Proof of Lemma ??. We know that the following limit exists for all $z \in Q \setminus K_N$:

$$0 < \lim_{n \to \infty} \frac{P_z(T_{K_N} > n)}{\underline{P}_0(T_{\{0\}} > n)} < \infty$$

This is a special case of the Kesten-Spitzer Ratio Limit Theorem, for more details, see [**?**], page 165. Let us denote the limit by c_z. Then we know that the series $\frac{P_z(T_{K_N} > n)}{\underline{P}_0(T_{\{0\}} > n)}$ converges to c_z for all $z \in H$. Thus

$$\liminf_{n \to \infty} \min_{z \in H} \frac{P_z(T_{K_N} > n)}{\underline{P}_0(T_{\{0\}} > n)} = \min_{z \in H} c_z > 0 \tag{2}$$

Given that $P_z(T_{K_N} > n) > 0$ for all z and n, the inequality (**??**) proves the statement of the lemma.

Proof of Lemma ??. For $d \geq 3$, the random walk is not recurrent, so the statement is trivial. For $d = 2$, in the case of finite second moments for $\underline{P}$, by the Local Central Limit Theorem ([**?**], page 25), the probability of return to the origin in the nth step is $u(n) = c_1/n + o(1/n)$, so the number of returns before the nth step

$$\theta_n = \sum_{k=0}^n u(k) = O(\log n).$$

If the distribution of the steps of the random walk is B-type, then we will use the Local Central Limit Theorem proven by Rvaceva, see Theorem 6.1 of [**?**] in the Appendix.

Our B-type case satisfies the requirements (aperiodicity is also assumed), so 2 dimensions $u(n) = \frac{c_1}{B_n^2} + o\left(\frac{1}{B_n^2}\right)$, where $c_1 = g(0)$ by the quoted theorem's notation, and

$$\theta_n = \sum_{k=0}^n u(k) = O(\log n)$$

It can also happen that the distribution of jump along one of the axes has finite second moment, whereas the other is B - type, then assuming they are independent, we can still easily show that $\theta_n = O(\log n)$.

Proof of Lemma ??. By the definition it follows that $\overline{\nu}_n \geq \nu_n$, so $E_z(\overline{\nu}_n) \geq E_z(\nu_n)$, and we only need to prove the inequality for $\overline{\nu}_n$.

Let us denote by S the series $X_0, X_1, X_2, \ldots,$, then create a new series $S' = \{\overline{X}_0, \overline{X}_1, \ldots\}$ by taking out every element of S that are in K_N. Then we define $\phi : \mathbb{Z}^+ \cup 0 \to \mathbb{Z}^+ \cup 0$ such that the index of $\overline{X}_i$ in the original series S is $\phi(i)$ (it is easy to see that $\phi(i) \geq i$). Finally let the set J be the following:

$$J = \{j : X_{\phi(j)+1} \in K_N\}$$

Then J is the set of j indexes such that we jump to K_N from the element corresponding to $\overline{X}_j$ in S. We define a complete system of events A_i, $i = 0, 1, \ldots n$:

$$A_i = \{i \in J, \{i+1, \ldots, n\} \text{ and } J \text{ are disjoint}\}$$

So the meaning of A_i is that from the first n steps outside K_N, the last step when we jump to K_N is the ith. A_0 corresponds to the event that we do not get to K_N in the first n steps outside K_N (thus $T_z(K_N) > n$). We denote the complement of K_N by $\overline{K_N}$. Then

$$1 = \sum_{i=0}^{n} P_z(A_i) = \sum_{i=0}^{n} \sum_{b \in \overline{K_N}} P_z(i \in J) P_b\left(S_{K_N} > n - i - 1\right) \cdot P_z(\overline{X}_{i+1} = b | i \in J) \quad (3)$$

Let K_N^* be the set of those $x \in Q \cap K_N$ points, from which we can jump out of K_N with positive probability. First we are going to deal with the case when there is such a $\tilde{b} \in (Q \setminus K_N)$ point that we can jump from every $x \in K_N^*$ to $\tilde{b}$ with positive probability (in 1 step). Then let $P_m = \min_{x \in K_N^*} P(x, \tilde{b})$, thus if we only take the tag of $\tilde{b}$ in (**??**), in the summation to b, (and by using that $P_b\left(S_{K_N} > n - i - 1\right) \geq P_b\left(S_{K_N} > n\right)$):

$$1 \geq \sum_{i=0}^{n} P_z(i \in J) P_{\tilde{b}}\left(S_{K_N} > n\right) \cdot P_z(\overline{X}_{i+1} = \tilde{b} | i \in J) \geq \sum_{i=0}^{n} P_z(i \in J) P_{\tilde{b}}\left(S_{K_N} > n\right) P_m$$

$$(4)$$

Moreover, $\sum_{i=0}^{n} P_z(i \in J) = E(\overline{\nu}_n)$, so by the choice $H = \{\tilde{b}\}$ and $D_{H,K_N} = 1/P_m$ the statement of the lemma is true (we have also used that $P_b(S_{K_N} > n)$ is independent of the transition probabilities inside K_N, so $\underline{P}_b(T_{K_N} > n) = P_b(S_{K_N} > n)$).

The second case is when no such $\tilde{b}$ point exists. Now for all $x \in K_N^*$, we choose such a $b_x \in (Q \setminus K_N)$ that we can jump there with positive probability, i.e. $P(x, b_x) > 0$. Let H be the set of these b_x points. It is evident from the construction that there is a constant $P_m > 0$ such that jumping out of K_N, we get to a point in H with greater or

equal probability than P_m ($P_m = \min\limits_{x \in K_N^*} P(x, b_x)$ is a good choice). Then

$$
\begin{aligned}
1 &= \sum_{i=0}^{n} \sum_{b \in \overline{K_N}} P_z(i \in J) P_b\left(S_{K_N} > n - i - 1\right) \cdot P_z(\overline{X}_{i+1} = b | i \in J) \\
&\geq \sum_{i=0}^{n} P_z(i \in J) \sum_{b \in H} P_b\left(S_{K_N} > n\right) \cdot P_z(\overline{X}_{i+1} = b | i \in J) \\
&\geq \sum_{i=0}^{n} P_z(i \in J) \cdot \min_{b \in H} P_b\left(S_{K_N} > n\right) \cdot \sum_{b \in H} P_z(\overline{X}_{i+1} = b | i \in J) \\
&= \sum_{i=0}^{n} P_z(i \in J) \cdot \min_{b \in H} P_b\left(S_{K_N} > n\right) \cdot P_z(\overline{X}_{i+1} \in H | i \in J) \\
&\geq \sum_{i=0}^{n} P_z(i \in J) \cdot \min_{b \in H} P_b\left(S_{K_N} > n\right) \cdot P_m
\end{aligned}
$$

As in the previous case, we get the statement of our lemma with $D_{H,K_N} = 1/P_m$.

Proof of Corollary ??. The outline of the proof is similar to page 355-356., [?], but here we prove it to random walk, instead of simple symmetric random walk. The idea to use this article comes from Péter Nándori.

First, we're going to suppose that the distribution has second moment. Then for $d = 2$, the probability of returning to the origin in the nth step is (by the Local Central Limit Theorem, [?], page 25):

$$
u(n) = 1/\left((2\pi)^{d/2} \sqrt{\det \Gamma} \cdot n^{d/2}\right) + o\left(1/n^{d/2}\right) = g/n^{d/2} + o\left(1/n^{d/2}\right) \tag{5}
$$

Here Γ is the covariance-matrix (size $d \times d$), and $g = 1/\left((2\pi)^{d/2} \sqrt{\det \Gamma}\right)$.

Denote by $R(n)$ the probability that the random walk does not return to the origin in n steps, then

$$
\sum_{k=0}^{n} u(k) R(n - k) = 1
$$

If $d \geq 3$, then

$$
U = \sum_{n=0}^{\infty} u(n) < \infty
$$

Let $R = \lim\limits_{n \to \infty} R(n)$, which exists because $R(n)$ is bounded and monotone. For $1 \leq k \leq n$

$$
R(n - k) \sum_{i=0}^{k} u(i) + \sum_{i=k+1}^{n} u(n) \geq 1
$$

If $k \to \infty$ and $n - k \to \infty$, then

$$
R(n - k) \cdot U \geq 1 + o(1)
$$

so

$$
R \geq \frac{1}{U}
$$

But $R(n)$ is monotone decreasing, so

$$R(n) \geq \frac{1}{U}.$$

If $d = 2$, then by substituting (**??**) in the place of $u(k)$, we get

$$u(0) + u(1) + \ldots u(n) = g \log n (1 + o(1))$$

using that $R(n)$ is decreasing

$$R(n) g \log n \leq 1 + o(1) \tag{6}$$

We know that for $0 < k < n$,

$$R(n - k)[u(0) + \ldots u(k)] + u(k + 1) + \ldots + u(n) \geq 1$$

If k and n tends to infinity, then

$$R(n - k) \cdot g \log(k)(1 + o(1)) + g(1 + o(1)) \log \frac{n}{k} \geq 1$$

Let $k = n - [n/\log n]$, then

$$R(n - k) g \log(n - k)(1 + o(1)) + o(1) \geq 1$$

This and (**??**) yields

$$R(n) = \frac{1 + o(1)}{g \log n}$$

If the distribution is B-type, then in d dimensions, according to the Local Central Limit Theorem in [**?**] (see Appendix), there is such a B_n series and $c = g(0)$ positive constant such that

$$u(n) = \frac{c}{B_n^d} + o\left(\frac{1}{B_n^d}\right)$$

We see that $\lim_{n \to \infty} \frac{B_n}{\sqrt{n}} > 0$. For $d = 2$, let us call $C_n = \sum_{k=0}^{n} u(k) = \sum_{k=0}^{n} \frac{c}{B_n^2}$, then $C_n = O(\log n)$ and it is monotone increasing.

Now, depending on whether $\lim_{n \to \infty} C_n = U < +\infty$ or $\lim_{n \to \infty} C_n = +\infty$, we have two cases. In the first case, using the same argument as before, we can show that $R(n) \geq \frac{1}{U}$. In the second one, we also use the same argument as previously: using the fact that $\sum_{k=0}^{n} u(k) R(n - k) = 1$, we can easily see that

$$R(n) C_n \leq 1 + o(1) \tag{7}$$

We know that for $0 < k < n$,

$$R(n - k)[u(0) + \ldots u(k)] + u(k + 1) + \ldots + u(n) \geq 1$$

If k and n tends to infinity, then

$$R(n - k) C_k (1 + o(1)) + (C_n - C_k) \geq 1$$

Let $k = n - [n/C_n]$, then

$$C_n - C_k \leq c(\log n - \log(n - [n/C_n]))(1 + o(1)) = c \log\left(\frac{1}{1 - 1/C_n}\right)(1 + o(1)) = o(1).$$

On the other hand, we see that

$$C_k - C_{n-k} \le (\log(n - [n/C_n]) - \log([n/C_n]))(1 + o(1)) =$$
$$= \log(C_n - 1)(1 + o(1)) = o(C_n) = o(C_k) = o(C_{n-k}),$$

therefore we can write

$$R(n - k)C_{n-k}(1 + o(1)) + o(1) \ge 1$$

This and (**??**) yields

$$R(n) = \frac{1 + o(1)}{C_n}$$

From these, Lemma **??**. and Lemma **??**., we get that

$$\overline{\lim_{n \to \infty}} \frac{E_z(\overline{\nu}_n)}{\underline{E}_0(\theta_n)} \le \lim_{n \to \infty} \frac{\underline{P}_0(T_{\{0\}} > n)}{\underline{P}_z(T_{K_N} > n)} < \infty$$

$$E_z(\overline{\nu}_n) = O(\log n)$$

Proof of Lemma ??.

Definition 11 Let us denote the time spent outside of K_N between the $i - 1$th and ith visit to K_N by η_{i-1}. This is the same as the length of the $i - 1$th zero block in the series $\chi_{x_0 \in K_N}, \chi_{x_1 \in K_N}, \ldots$. Let ξ_{i-1} be the time spent in K_N by X_N during the ith visit, which equals the length of the ith ones block in the previous series.

The difficulty in the proof is that ξ_i are dependent, so they are hard to handle directly. For this reason, we're going to work with the conditional probabilities in function of the entrance/exit points.

Definition 12 Let u_k be the point of the kth entrance to K_N ($u_k \in K_N$), and v_k the last point in K_N before the kth exit from K_N. Let us denote $U_n = \{u_k\}_{k=1}^n$ and $V_n = \{v_k\}_{k=1}^n$. Then U_n and V_n are random vectors taking values in K_N^n.

There exists such a $K < \infty$ that $E_z(\xi_k|u_k, v_k) \le K$ because the expected value only depends on the value of u_k and v_k (but does not depends on k), and there are at most $|K_N|^2$ u, v pairs, and the expected value is bounded for all of them, so we can choose K as the maximum of these. In the following, we'll use the shorthand notation for summation $\sum_{U_n, V_n} \overset{def}{=} \sum_{U_n \in K_N^n, V_n \in K_N^n}$,

$$E_z(\rho_n) \le E_z(\sum_{i=1}^{\overline{\nu}_n} \xi_i) = \sum_{k=0}^n E(\sum_{i=1}^k \xi_i | \overline{\nu}_n = k) P(\overline{\nu}_n = k) =$$

$$= \sum_{k=0}^n \sum_{i=1}^k \sum_{U_n, V_n} E(\xi_i | \overline{\nu}_n = k, u_i(U_n), v_i(U_n)) P(\overline{\nu}_n = k, U_n, V_n) \le$$

$$\le \sum_{k=0}^n \sum_{U_n, V_n} K \cdot k \cdot P(U_n, V_n, \overline{\nu}_n = k) = K \sum_{k=0}^n k \cdot P(\overline{\nu}_n = k) = K \cdot E(\overline{\nu}_n)$$

Here we have used that $E(\xi_i|u_i, v_i, \overline{\nu}_n = k) = E(\xi_i|u_i, v_i) \le K$, this is true because $\overline{\nu}_n$ is independent of the time spent inside K_N. Now it is clear that $E_z(\rho_n) = O(\log n)$.

6 Remarks

1. This paper is based on the work done for D. Paulin's bachelor's thesis in 2009 at Budapest University of Technology and Economics.
2. The conditions for our Theorem 1 are quite general. With some technical work, it could be easily shown that Theorem 1 also holds for periodic random walks, the same way it is done in [?].
3. In relation to Theorem 2, we conjecture that, analogously to the finite horizon case, the weak limit of the locally perturbed Lorentz-process with infinite horizon is the same Brownian motion as it is that of the periodic one.
4. If for the distribution of the jumps in the random walk,

$$\sup_{k \in \mathbb{R}^+} (k. \text{ moment exists}) = \alpha,$$

with $\alpha \in [1, 2)$, and the tail of the random walk satisfies some criteria (see [?] page 114 Theorem 4.5.1), then we will converge to a stable distribution with parameter α, and the normalizing factor will not be $n^{1/2}$, but $n^{1/\alpha} L_0(n)$, L_0 being a slowly varying function. We can apply the local limit theorem for stable distributions, Theorem 6.1 of [?], to the returns to the origin, $P(Y_n^{(1)} = 0) = O(n^{-1/\alpha})$. Using the same theorem we get that in two dimensions, if the jump belongs to the domain of attraction of the 2 dimensional stable law with parameter α, then $P\left(Y_n^{(2)} = 0\right) = O(n^{-2/\alpha})$. $\alpha \in [1, 2)$, so $\sum_{n=0}^{\infty} P\left(Y_n^{(2)} = 0\right) < \infty$, and thus the random walk is transient. From this, the convergence to a Lévy - process can be proven.

7 Appendix

In what follows we summarize some notions and theorems related to multidimensional limit theory of sums of iid random vectors, in particular some global and local limit theorems and domains of attractions. These are typically not included in textbooks on probability, and moreover, the pioneering work of Rvaceva on local theorems is not easily available.

Definition 13 Let $\{\xi(n)\}_n$ be a sequence of iid random vectors, with distribution function $F(x)$. Then if there are suitably chosen constants $C(n) > 0$, real vectors $d(n)$ such that $s_n = \sum_{k=1}^{n} \xi_k / C(n) - d(n)$ converges in distribution to a non-degenerate probability distribution $R(x)$, then

1. R is called a stable law;
2. and we say that $F(x)$ belongs to the domain of attraction of $R(x)$.

Our main interest in this paper is the domain of attraction of the gaussian law, therefore below we also restrict our attention to it.

Theorem 4.1 of [?] describes the domain of attraction of the normal law for random vectors (below ' means matrix transpose):

Theorem 4.1([**?**]). P belongs to the domain of attraction of the non-degenerate normal law with characteristic function $\exp(-Q(t)/2)$ if and only if:

$$(1) \qquad R^2 \int_{|x|>R} \mathrm{d}P(x) / \int_{|x|<R} x'x \, \mathrm{d}P(x) \to 0 \text{ as } R \to \infty,$$

$$(2) \qquad \int_{|x|<R} (t'x)^2 \, \mathrm{d}P(x) / \int_{|x|<R} (u'x)^2 \, \mathrm{d}P(x) \to Q(t)/Q(u) \text{ as } R \to \infty,$$

for arbitrary $t, u \in \mathbb{R}^d$.

Further if the jump distribution $\underline{P}$ of the random walk is B-type, then we define $\eta_n^{Y'}(t) = B_n^{-1} Y_{nt}$ for $t = 0, \frac{1}{n}, \frac{2}{n}, \ldots, 1$ and take its piecewise linear extension. It is well-known that $\eta_n^{Y'} \Rightarrow W_{\sum}(t)$, a d-dimensional Wiener process (in fact, Skorohod proved that when the CLT holds, then this functional central limit theorem holds too; for more details, see [**?**] and [**?**], page 115-118). The covariance matrix $\sum$ is determined by Theorem 4.1 of [**?**].

Definition 14 As a special case, we call a B-type random variable (or random vector) L-type if $B_n = \sqrt{cn \log n}$. (N. B.: this scaling is used in the weak limit of the planar, infinite-horizon Lorentz-process; cf. [**?**] and [**?**]). Now $L(x) \sim 2c \log x$. If we denote by S_n^* the sum of n iid one dimensional L-type variables, then $\frac{S_n^*}{\sqrt{cn \log n}} \xrightarrow{d} N(0, 1)$.

Finally we recall Rvaceva's Local Limit Theorem, Theorem 6.1 of [**?**]:

Let $\{\xi(n)\}_n$ be a sequence of iid $\mathbb{Z}^p$-valued random vectors and let $P(x) = Pr(\xi(n) = x)$. Let $P(n; z) = Pr(s(n) = z)$, where $s(n) = \xi(1) + \ldots + \xi(n)$, and $g(x)$ be the density of a certain stable distribution G.

Theorem 6.1 of [?] In order that for some suitably chosen constant vectors $a(n)$ and positive constants $B(n)$ the relation

$$R(n) = B^p(n) P(n; z) - g[[z - a(n)]/B(n)] \to 0$$

hold uniformly with respect to z, it is necessary and sufficient that the distribution of $\xi(n)$

1) belong to the domain of attraction of G, and
2) be a 1 - lattice distribution
   The second condition is equivalent to each of the following:
2') The greatest common divisor of the volumes of p - dimensional simplexes a $p + 1$ vertices of which lie at points with $P(x) > 0$ is $1/p!$
2") the lattice generated by all vectors $(x - y)$ such that $P(x) > 0 < P(y)$ coincides with the lattice of all integral points of the p - dimensional space

Remark 5 The series $B(n)$ in this theorem may be different from $C(n)$ in Definition **??**. If a two dimensional RW jumps independently along the axes, and it has a distribution with finite second moment along one axis and an L-type distribution along the other, then $C(n) \sim \sqrt{n \log n}$ and $B(n) \sim \sqrt{n} \log^{1/4} n$.

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