CALCULATIONS FOR $d = 2$
(Á LA CHERNOV-MARKARIAN).

DOMOKOS SZÁSZ

0.1. **Coordinate representation of the flow.** $d = 2$. Coordinates: $z = (x, y, \omega) \in M = Q \times S_1$, where $q = (x, y) \in Q$, e_x, e_y unit coordinate vectors, $\omega = \angle(v, e_x)$. Flow $S^t : (x^-, y^-, \omega^-) \to (x^+, y^+, \omega^+)$. Task: DS^t.

Free motion:

$$x^+ = x^- + t\cos\omega \qquad y^+ = y^- + \sin\omega \qquad \omega^+ = \omega^-$$

Thus

$$DS^t = \begin{pmatrix} 1 & 0 & -t\sin\omega^- \\ 0 & 1 & t\cos\omega^- \\ 0 & 0 & 1 \end{pmatrix}$$

Collision: Assume one regular collision in time interval $(0, t)$. Denote: $(\bar{x}, \bar{y}) \in \partial Q_i$ the collision point, $\mathcal{T}$ tangent vector to the boundary, $\gamma = \angle(\mathcal{T}, e_x)$, s^--collision time, $\psi = \angle(v^+, \mathcal{T})$. Trivial equations:

$$x^- = \bar{x} - s^-\cos\omega^- \qquad x^+ = \bar{x} + s^+\cos\omega^+ \tag{1}$$

$$y^- = \bar{y} - s^-\sin\omega^- \qquad y^+ = \bar{y} + s^+\sin\omega^+ \tag{2}$$

$$\omega^- = \gamma - \psi \qquad \omega^+ = \gamma + \psi \tag{3}$$

Denote: r the arc-length parameter on ∂Q_i. Then

$$d\bar{x} = \cos\gamma dr \qquad d\bar{y} = \sin\gamma dr \qquad d\gamma = -\mathcal{K}dr$$

where $\mathcal{K}$ is the curvature of the boundary. The derivatives of (1- 3) are

$$dx^+ = \cos\gamma dr + \cos\omega^+ ds^+ - s^+\sin\omega^+ d\omega^+ \tag{4}$$

$$dy^+ = \sin\gamma dr + \sin\omega^+ ds^+ + s^+\cos\omega^+ d\omega^+ \tag{5}$$

$$d\omega^+ = -\mathcal{K}dr + d\psi \tag{6}$$

and

$$dx^- = \cos\gamma dr - \cos\omega^- ds^- + s^-\sin\omega^- d\omega^- \tag{7}$$

$$dy^- = \sin\gamma dr - \sin\omega^- ds^- - s^- \cos\omega^- d\omega^- \tag{8}$$

$$d\omega^- = -\mathcal{K}dr - d\psi \tag{9}$$

These equation permit to calculate the jacobians $\frac{\partial(x^-,y^-,\omega^-)}{\partial(r,s,\psi)} = -\sin\psi$, $\frac{\partial(x^+,y^+,\omega^+)}{\partial(r,s,\psi)} = \sin\psi$

Theorem 1. *The Lebesgue-measure $dx\,dy\,d\omega$ is invariant wrt the flow S^t.*

0.2. **Derivative of the Poincaré section map.** $d = 2$. Let $z = (r, \phi), z_1 = (r_1, \phi_1) \in \partial M = \partial Q \times [-\pi/2, \pi/2]$ the phase points at two subsequent collisions. On the boundary curves we use arc-length parametrization, whereas ϕ and ϕ_1 denote the angles of the incoming (and, by the reflection law, also outgoing) velocities with the normal vectors n and n_1 to the boundary, pointing inside the billiard table (if $\phi = \pm\pi/2$, then we are calculating the one-sided derivatives, never mind). Tangent vectors to the boundary pieces at the collision points will be denoted by $\mathcal{T}, \mathcal{T}_1$, their angles with the x-axis by γ, γ_1 and with the outgoing velocities by ψ and ψ_1, the curvatures of the boundaries by $\mathcal{K}, \mathcal{K}_1$. Further notations: $||z_1 - z|| = \tau$ is the free path length, ω is the angle between the x-axis and the vector $z_1 - z$, $z = (x, y), z_1 = (x_1, y_1)$.

By the notations

$$x_1 - x = \tau\cos\omega, \qquad y_1 - y = \tau\sin\omega \tag{10}$$

$$\phi + \psi = \pi/2, \qquad \phi_1 + \psi_1 = \pi/2 \tag{11}$$

$$dx = \cos\gamma dr, \qquad dy = \sin\gamma dr, \qquad d\gamma = -\mathcal{K}dr \tag{12}$$

$$dx_1 = \cos\gamma_1 dr_1, \qquad dy_1 = \sin\gamma_1 dr_1, \qquad d\gamma_1 = -\mathcal{K}_1 dr_1 \tag{13}$$

$$\omega = \gamma + \psi = \gamma_1 - \psi_1 \tag{14}$$

By differentiating (14)

$$d\omega = -\mathcal{K}dr + d\psi = -\mathcal{K}_1 dr_1 - d\psi_1 \tag{15}$$

and by differentiating (10)

$$\cos\gamma_1 dr_1 + \cos\gamma dr = \cos\omega d\tau - \tau\sin\omega d\omega \tag{16}$$

$$\sin\gamma_1 dr_1 - \sin\gamma dr = \sin\omega d\tau + \tau\cos\omega d\omega \tag{17}$$

Eliminate $d\tau$ from (16-17)

(18)
$$dr_1(-\sin\omega\cos\gamma_1 + \cos\omega\sin\gamma_1) + dr(\sin\omega\cos\gamma - \cos\omega\sin\gamma) =$$

(19) $$dr_1\sin(\gamma_1 - \omega) + dr\sin(\omega - \gamma) = dr_1\sin\psi_1 + dr\sin\psi = \tau d\omega$$

Solve (15) and (19) for $dr_1, d\phi_1$ by also using (11) (note: $\sin\psi_1 = \cos\phi_1, \sin\psi = \cos\phi$). From (18-19) and (15)

(20) $$dr_1 = \frac{-1}{\cos\phi_1}\left[dr(\cos\phi + \tau\mathcal{K}) + \tau d\phi\right]$$

and from (15)

(21) $$d\phi_1 = -d\psi_1 = \mathcal{K}_1 dr_1 - \mathcal{K}dr - d\phi$$

Now the desired Jacobian is

(22)
$$\frac{\partial(r_1,\phi_1)}{\partial(r,\phi)} = \frac{-1}{\cos\phi_1}\begin{pmatrix} \cos\phi + \tau\mathcal{K} & \tau \\ \mathcal{K}_1\cos\phi + \tau\mathcal{K}\mathcal{K}_1 + \mathcal{K}\cos\phi_1 & \mathcal{K}_1\tau + \cos\phi_1 \end{pmatrix}$$

Theorem 2. *If the boundary pieces are C^l-smooth then the map $T : \tilde{M}(= \partial Q \times (S_1)_+) \setminus \mathcal{S}_1 \to \tilde{M} \setminus S_{-1}$ is a C^{l-1}-diffeo.*

Proof: DT is expressed though curvatures, i. e. second derivatives.

Theorem 3. *The map T preserves the measure* $\cos\phi\; dr\; d\phi$ *on $\tilde{M}$.*

0.3. **Jacobi coordinates.** Introduce new coordinates $(d\eta, d\xi, d\omega)$, the so-called Jacobi-coordinates, in $\mathcal{T}_{\mathbf{x}}M$:

$$d\eta = \cos\omega\; dx + \sin\omega\; dy \qquad d\xi = -\sin\omega\; dx + \cos\omega\; dy$$

For $dZ = (d\eta, d\xi, d\omega)$ calculate $D_Z S^t(dZ) = dZ_t = (d\eta_t, d\xi_t, d\omega_t)$. Of course,

$$D_{\mathbf{x}}S^t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$$

thus it is sufficient to calculate $D_{\mathbf{x}}S^t$ in the subspace $\mathcal{T}_Z^{\perp}M$.

Free motion.

Theorem 4. *For the free motion the derivative map $D_Z S^t$ on $\mathcal{T}_Z^{\perp}M$ is given by the upper triangular matrix*

$$U_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

Collision. Assume that the trajectory segment contains exactly one (regular) collision. From (4-6) and (7-9) one has

$$d\xi^- = \cos\phi dr - s^- d\omega^- \qquad d\omega^- = -\mathcal{K} \quad dr + d\phi$$
$$d\xi^+ = -\cos\phi dr + s^+ d\omega^+ \qquad d\omega^+ = -\mathcal{K} \quad dr - d\phi$$

(by $\psi + \phi = \pi/2$ one has $d\psi = -d\phi$). Put $s^- = s^+ = 0$ and eliminate dr and $d\phi$ to obtain

$$d\xi^+ = -d\xi^- \tag{23}$$
$$d\omega^+ = -\mathcal{R} d\xi^- - d\omega^- \tag{24}$$

where $\mathcal{R} = \frac{2\mathcal{K}}{\cos\phi}$.

Theorem 5. *Assume that the trajectory segment contains exactly one (regular) collision. Then the non-trivial* 2×2 *minor of the derivative is the lower triangular matrix*

$$L_{\mathcal{R}} = \begin{pmatrix} 1 & 0 \\ \mathcal{R} & 1 \end{pmatrix}$$

N. B. The sign changes as compared to (23-24) are the results of the change of orientation caused by the reflection.

Corollary In general, the derivative map $D_Z S^t$ on $\mathcal{T}_Z^\perp M$ is the product of alternating upper and lower triangular matrices

$$D_Z S^t = (-1)^n U_{t-t_n} \, L_{\mathcal{R}_n} \, U_{t_n - t_{n-1}} \, L_{\mathcal{R}_{n-1}} \dots U_{t_2 - t_1} L_{\mathcal{R}_1} \, U_{t_1}$$

Theorem 6. *Let* $L \subset \mathcal{T}_Z^\perp M$ *be a tangent line with slope* $\mathcal{B}$*. Let* $L_t = D_Z S^t(L) \subset \mathcal{T}_{S^t Z}^\perp M$ *be its image at time t and let* $\mathcal{B}_t$ *be its slope. Then*

- *if the interval* $[0, t]$ *does not contain collisions, then*
$$\mathcal{B}_t = \frac{\mathcal{B}}{1 + t\mathcal{B}} = \frac{1}{1 + \frac{1}{B}}$$
- *if t is a collision moment, then the pre-collisional and post-collisional slopes* $\mathcal{B}^-$ *and* $\mathcal{B}^+$ *are related by*
$$\mathcal{B}^+ = \mathcal{R} + \mathcal{B}^- \qquad \mathcal{R} = \frac{2\mathcal{K}}{\cos\phi}$$

Theorem 7. *In general, the slope* $\mathcal{B}_t$ *is*

$$\mathcal{B}_t = \cfrac{1}{t - t_n + \cfrac{1}{\mathcal{R}_n + \cfrac{1}{t_n - t_{n-1} + \cfrac{1}{\mathcal{R}_{n-1}} + \cfrac{1}{\dots + \cfrac{1}{t_1 + \frac{1}{B}}}}}}$$