EXISTENCE OF LOCAL INVARIANT MANIFOLDS FOR THE SINGULAR CAT.

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Let $T_A \equiv Ax \pmod{\mathbb{Z}^2}$ an algebraic auto of $\mathbb{T}^2$ with invariant measure $\mu = Leb$. Here, for simplicity, $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Its singular version T is an endo of the unit square $Q = [0,1]^2$ with the same invariant measure. (By a simple argument the ergodicity of T is a direct consequence of that of T_A.) Denote $S_n = T^n \partial Q : n \in \mathbb{Z}$.

Facts:

(1) the set of discontinuities of T is S_{-1};
(2) for $n \geq 1$, the set of discontinuities of T^n is $\cup_{l=1}^{l=n} S_{-l}$;
(3) for $n \geq 1$, the set of discontinuities of T^{-n} is $\cup_{l=1}^{l=n} S_l$.

Denote $\mathcal{SR}_+ = \cup_{l=0}^{\infty} S_l$ and $\mathcal{SR}_- = \cup_{l=0}^{\infty} S_{-l}$. (Note: T is an auto on $Q \setminus (SR_- \cup SR_+)$.) Reminder:

$$\gamma^{s(u)}(x) = \{y | \lim_{n \to \infty} dist(T^{n(-n)}x, T^{n(-n)}y) = 0\}$$

therefore

$$\gamma^{s(u)}(x) = \{x + te^{s(u)} | t \in \mathbb{R}\}$$

Definition 1. *Consider the subinterval* $[x, y] \in \gamma^s(x)$.

$\gamma^s_{loc}(x) := \{y \in \gamma^s(x) | \forall n \geq 0$ *the interval* $T^n[x, y]$ *never intersects* $\mathcal{SR}_-\}$

or in other words

$$\gamma^s_{loc}(x) := \{y \in \gamma^s(x) | \forall n \geq 0 \ T^n \textit{ is continuous on } [x, y]\}$$

Definition 2. *Consider the subinterval* $[x, y] \in \gamma^s(x)$.

$\gamma^u_{loc}(x) := \{y \in \gamma^u(x) | \forall n \leq 0$ *the interval* $T^n[x, y]$ *never intersects* $\mathcal{SR}_+\}$

or in other words

$$\gamma^u_{loc}(x) := \{y \in \gamma^u(x) | \forall n \leq 0 \ T^{-n} \textit{ is continuous on } [x, y]\}$$

Theorem 1. *(***Existence of local invariant manifolds***.) For a. e.* $x \in Q$, $\exists \delta(x) > 0$, *s. t.* $|\gamma^u_{loc}(x)|, |\gamma^s_{loc}(x)| > 2\delta(x)$ *holds.*

Proof. Denote length measured along γ^u_{loc} by d^u and the unstable eigenvalue of A by $\lambda^u > 1$. Let $\alpha > 1$ and $c > 0$. By simple geometry,

$$\mu\{x|d^u(x, S_0) \le \varepsilon\} \le const.\varepsilon. \tag{1}$$

By Definition 2 we need:

$$\textit{for a. e. } x \in Q\ \exists\delta(x) > 0,\ s.\ t. \forall n \ge 0\ d^u(x, S_n) > \delta(x)$$

i. e.

$$\textit{for a. e. } x \in Q\ \exists\delta(x) > 0,\ s.\ t. \forall n \ge 0\ d^u(T^{-n}x, S_0) > \frac{\delta(x)}{(\lambda^u)^n}$$

By (**??**) and measure preserving

$$\sum_{n\ge 0} \mu(d^u(x, S_0) \le \frac{c}{n^\alpha}) = \sum_{n\ge 0} \mu(d^u(T^{-n}x, S_0) \le \frac{c}{n^\alpha}) < \infty$$

thus, by Borel-Cantelli, for a. e. point x – apart from a finite set of indices $n \ge 0$ –

$$d^u(T^{-n}x, S_0) > \frac{c}{n^\alpha}.$$

Consequently for a. e. x

$$\exists c(x) > 0\ \ s.\ t.\ \forall n \ge 0\ \ d^u(T^{-n}x, S_0) > \frac{c(x)}{n^\alpha}$$

or equivalently

$$\forall n \ge 0\ d^u(x, S_n) > (\lambda^u)^n \frac{c(x)}{n^\alpha}$$

Let $\delta(x) := \min_{n\ge 0}((\lambda^u)^n \frac{c(x)}{n^\alpha}) > 0$. Now it is easy to see that the theorem holds with this function $\delta(x)$.

Remark. The proof, of course, works in the same way, for any hyperbolic algebraic auto of $\mathbb{T}^d : d > 2$. For $d > 2$, the linear map A may have eigenvalues outside the unit circumference in conjugate pairs. Then one uses the eigenvalues of $(AA^*)^{1/2}$. In the general case of hyperbolic diffeos of a compact Riemannian manifold with an invariant measure one uses Oseledec' theorem for defining the real Lyapunov exponents $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_{d^s} < 0 < \lambda_{d^s+1} \le \cdots \le \lambda_{d^s+d^u}$, and then the definition of the invariant manifolds $\gamma^u(x), \gamma^s(x)$ is itself non-trivial. However, if a hyperbolic diffeo of a compact Riemannian manifold with an invariant measure – like a semi-dispersing billiard – has (not too pathological) singularities, then under rather general conditions the above procedure is still applicable to show that local invariant manifolds exist almost everywhere.

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