Probability 2.
4. Exercise sheet
2010.04.01.

Applications of generating functions: Random Walks and Branching Processes

1. $X_1, X_2, X_3, \ldots$ are i.i.d. random variables, with common distribution function $F(x) := \mathbf{P}\big(X_i < x\big)$. Let ν be independent of these, $\mathbb{Z}_+$-valued random variable; and let us denote the generating function of ν by $G(z)$! Show that the random variable $Y := \max\{X_1, X_2, \ldots, X_\nu\}$ has distribution function $H(x) = G(F(x))$.

2. The probability that there are n maybugs in the vegetable garden of Steve is denoted by g_n. The generating function of the sequence g_n is denoted by $P(z)$. While hoeing, Steve kills each maybug (by cutting them in two...) independently with probability p. What is the generating function, the expected value and the variance of the number of surviving maybugs?

3. Let $X_0 = 1, X_1, X_2, \ldots$ be a Galton–Watson process, $\mathbf{E}(X_1) = \mu$, $\mathbf{D}^2(X_1) = \sigma^2$. Show that

   (a)
   $$\mathbf{D}^2(X_{n+1}) = \mu \mathbf{D}^2(X_n) + \mu^{2n}\sigma^2$$

   (b)
   $$\mathbf{D}^2(X_n) = \sigma^2 \mu^{n-1} \frac{1-\mu^n}{1-\mu}!$$

4. (Continue). Show that for $n > m$, $\mathbf{E}(X_n X_m) = \mu^{n-m}\mathbf{E}X_m^2$!

5. What is the probability that the Galton-Watson process survives for n generations, if the offspring distribution is pessimistic GEO(1/2)?

6. Let us consider a Galton-Watson process whose offspring distribution has generating function $\mathbf{E}(z^{X_1}) = P(z)$. Y denotes the size of the whole population (i.e. $Y = \infty$ if the process survives forever), and $Q(z) := \mathbf{E}(z^Y)$. Show that $Q(z)$ is the inverse function of $\frac{z}{P(z)}$! (Hint: condition on the value X_1!)

7. Let $S_0 = 0, S_1, S_2, \ldots$ be *Simple Symmetric Random Walk* (SSRW) on $\mathbb{Z}$, i.e. $S_n = X_1 + \ldots + X_n$, where $X_1, X_2, \ldots$ are i.i.d. of $\mathbf{P}(X_i = \pm 1) = \frac{1}{2}$. Furthermore, let us denote by
   $$\begin{aligned} &u_n = \mathbf{P}(S_n = 0), \quad (n = 0, 1, 2, \ldots), \\ &f_n = \mathbf{P}(S_1, \ldots, S_{n-1} \neq 0, S_n = 0), \quad (n = 1, 2, \ldots), \\ &f_0 = 0, \end{aligned}$$
   and the corresponding generating functions $U(z)$ and $F(z)$, respectively.

   (a) Show that $U(z) = (1 - z^2)^{-\frac{1}{2}}$!

   (b) Prove that
   $$U(z) = \frac{1}{1 - F(z)}.$$
   (Hint: notice that for $n \geq 1$ $u_n = f_1 u_{n-1} + f_2 u_{n-2} + \ldots f_n u_0$. why is this true?)

8. (a) (SSRW Continued) S_n is a SSRW on $\mathbb{Z}$ starting from the origin, τ denotes the first hitting time of 1, i.e. $\tau = \min\{n | S_n = 1\}$. Determine the exact asymptotics of τ:
   $$\limsup_{k \to \infty} k^{\frac{3}{2}} \mathbf{P}(\tau = k) = ?$$

Hint: write the series expansion for the generating function of τ, determine $P(\tau = 2k)$ and use Stirling-formula.

(b) $X_1, X_2, \ldots$ is a branching process with pessimistic $GEO(1/2)$ offspring disribution. The size of the total population is denoted by Y, (and has generating function $Q(z)$).

$$\lim_{k\to\infty} k^{\frac{3}{2}} \mathbf{P}(Y = k) =?$$

How can you prove this without any calculations, using the previous exercise? (*Hint:* try depth-first-search and probabilistic interpretation)

(c) An amoeba can do two things every day: independently of everything else (other amoebas and other days): splits into two or dies with probability $\frac{1}{2}$, respectively. (So the offspring distribution is 0 or 2, with equal probs.) The total number of vertices in the tree of the branching process are denoted by Y. Show that $Y \sim \tau$ (the first hitting time of 1 in a SSRW) using generating functions. Also give a probabilistic interpretation of this identity. (*Hint:* try breadth-first-search algorithm)

9. Given the generating function $P(z)$ of the random variable X, calculate the moments $\mathbf{E}(X^3)$ and $\mathbf{E}(X^4)$! Apply the result for a POI(λ) distributed random variable X.

10. Let us denote by X and Y the total number of 1-s and 6-s in a sequence of dice-throws of an unfair dice. Determine the joint generating function $H(z, w) := \mathbf{E}(z^X w^Y)$! (Use the notation $\mathbf{P}(Z = 1) = p_1$, $\mathbf{P}(Z = 6) = p_6$ for a single throw Z!)

11. (Continue.) Solve the previous exercise with the following modification: the number of throws is a POI(λ) distributed random variable N, independent of the throws. What is the joint distribution of X and Y in this case?

1. Galton-Watson, continued. Show that for $n > m$, the joint generating function of X_m and X_n is

$$P(z_1, z_2) = P_m(z_1 P_{n-m}(z_2)),$$

where P_i is the generating function of X_i.

2. Suppose that every Amway-agent ropes in 0, 1, and 2 new agents with prob. $\frac{1}{3}$ every day into the Amway-network, (independently of other agents and other days). Aunty Mary starts organizing its subnetwork on the 1^{st} of April. What is the expectation and variance of the total number of agents in her subnetwork (including herself as well) on the 30^{th} of April?

3. Let $X_1, X_2, \ldots$ be i.i.d. N^+ valued random variables with generating function $P(z)$, and $S_n := X_1 + X_2 + \ldots + X_n$. Let us define

$$Y_k = \begin{cases} 1, & \text{if } \exists n, \text{ such that } S_n = k, \\ 0 & \text{otherwise} \end{cases}$$

Let $v_k := \mathbf{P}(Y_k = 1)$, and its generating function $V(z) = \sum_{k=0}^{\infty} v_k z^k$. Show that $V(z) = \frac{1}{1-P(z)}$!

4. Now consider a Simple Asymmetric Random Walk on $\mathbb{Z}$: $p > q = 1 - p$, where p is the probability of jumping to the right. Let $\beta_n = \mathbf{P}(\exists j > n : S_j = 0)$. Show that the generating function of the sequence β_n is

$$\beta(z) = \frac{(1 - 4pqz^2)^{1/2} - (p - q)}{(1 - 4pqz^2)^{1/2}(1 - z)}!$$

1. The $2n$ vertices of an equilateral $2n$-polygon is paired randomly.

   (a) What is the probability p_n of the event, that the chords connecting the pairs inside the polygon do not intersect?

   (b) What is the convergence radius of the generating function of the sequence $\{p_n\}$?

   (c) What is the answer, if the polygon is convex but not equilateral?

   (d) How can the problem be generalized for convex polyhedra?