Local Limit of the Random Degree Constrained Process

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Joint work with Balázs Ráth and Lutz Warnke

1. Local limit of the RDCP
2. Critical time

Random Degree Constrained Process

RDCP on the complete graph:

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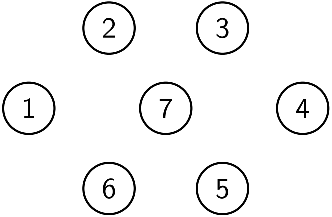
- Vertex v is **saturated** at time t if its degree in $G_{\underline{p}}^n(t)$ is equal to $d(v)$

An example for RDCP

RDCP on K_7 with degree constraint 3 for each vertex:



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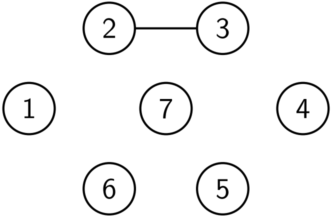

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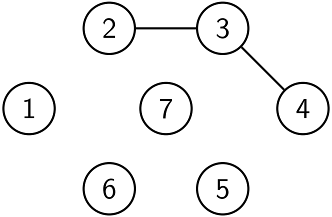

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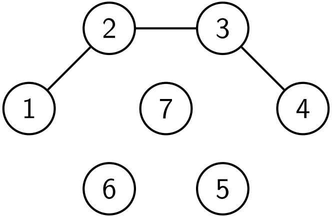

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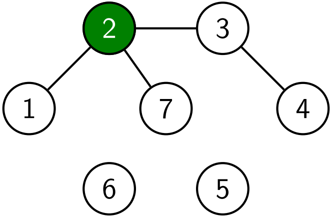

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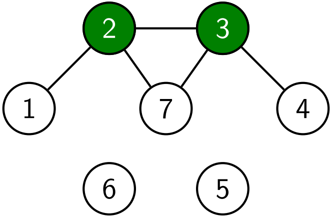

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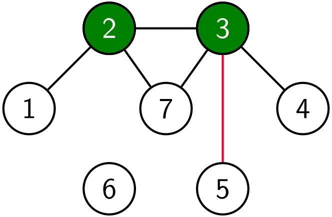

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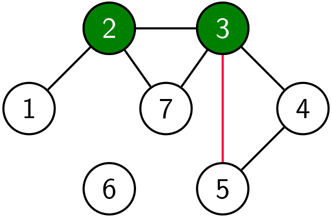

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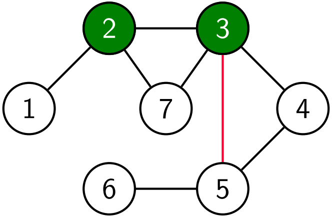

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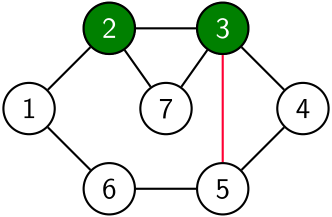

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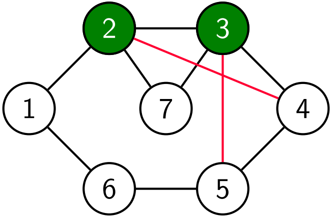

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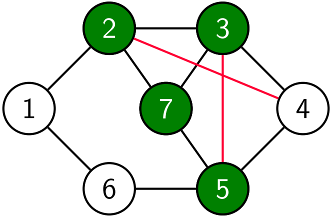

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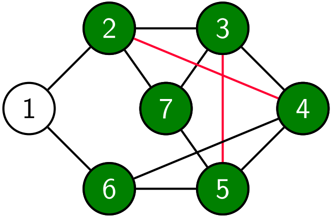

Configuration model

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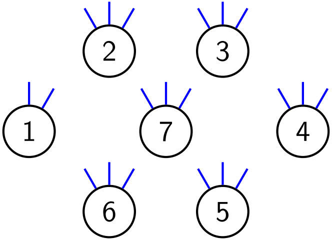

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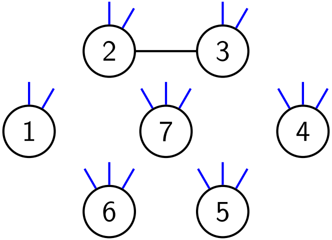

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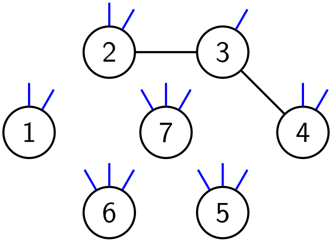

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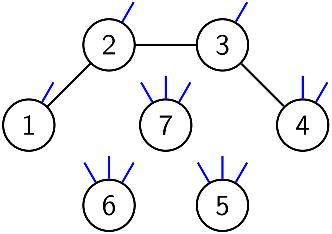

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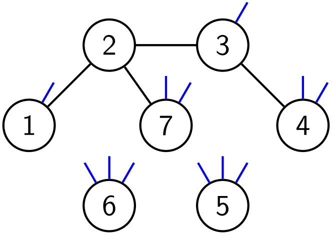

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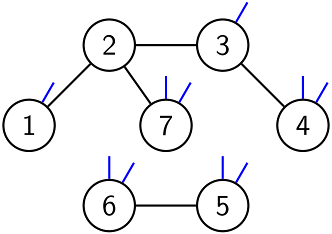

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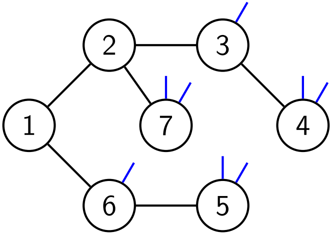

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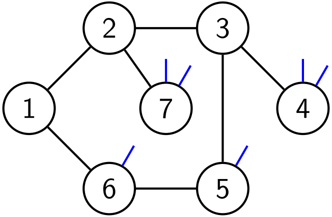

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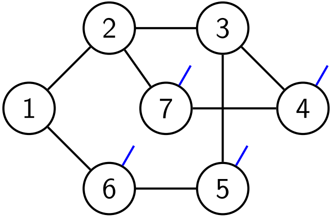

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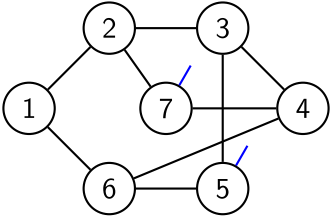

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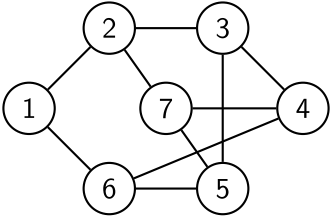

Comparison of the two models

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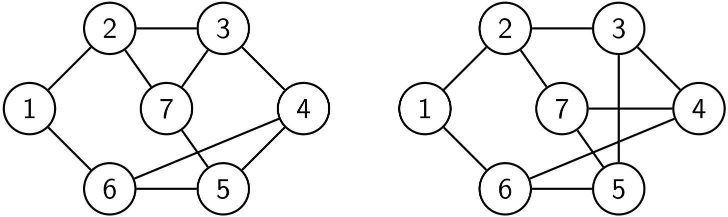

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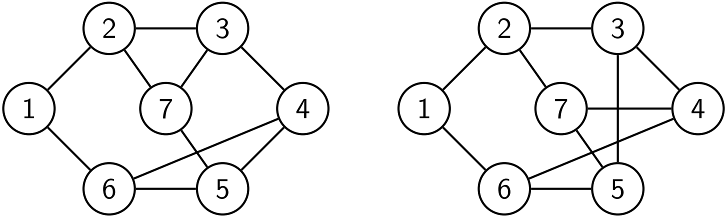

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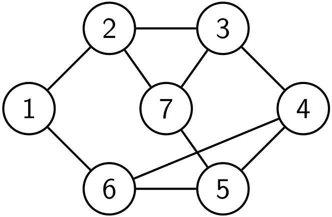

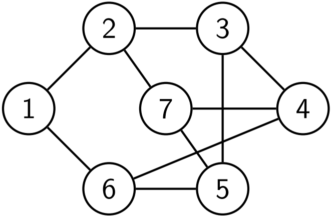

- Question: Is RDCP at time t a configuration model? No!

Theorem (RDCP is far from uniform (Molloy, Surya, Warnke, 2022+))

If the degree constraints are not nearly regular then the final graph of the RDCP is not contiguous to the configuration model with the same degree sequence.

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where $\mathcal{B}^R(u)$ is the R-neighbourhood of the root u in $\mathcal{G}$.

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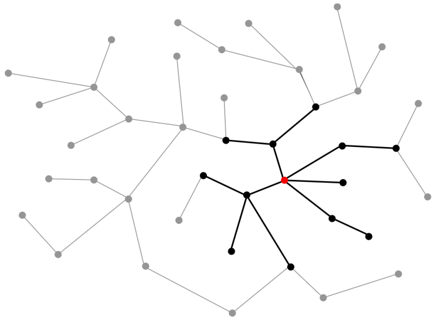

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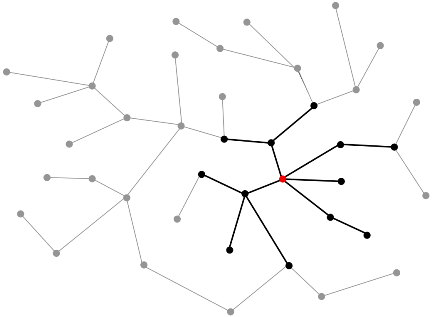

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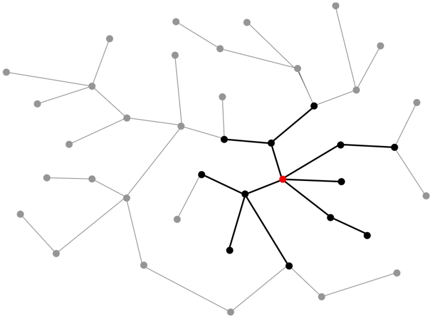

- It can be used to investigate numerous graph properties, parameters
- The limiting objects (often branching processes) are much simpler to study than the original model

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In contrast to the configuration model, the **local weak limit of the RDCP was unknown** before our work

Local weak limit of the RDCP

Theorem (Local weak convergence of RDCP (Ráth, Sz., Warnke, 2024+))

Fix $t \in \mathbb{R}_+ \cup \{\infty\}$. Then the random degree constrained process on K_n at time t converges locally to a random rooted tree $\left(G^{\infty}_{\underline{p}}(t), \varnothing\right)$, i.e.,

$$G^n_{\underline{p}}(t) \longrightarrow \left(G^{\infty}_{\underline{p}}(t), \varnothing\right) \text{ locally as } n \to \infty. \tag{2}$$

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We even have a more general result:

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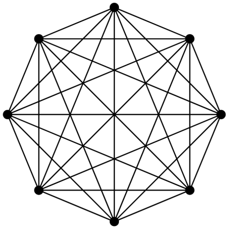

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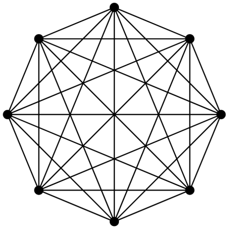

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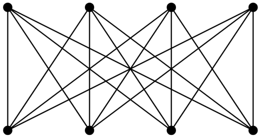

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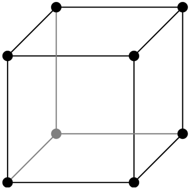

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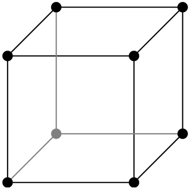

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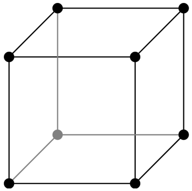

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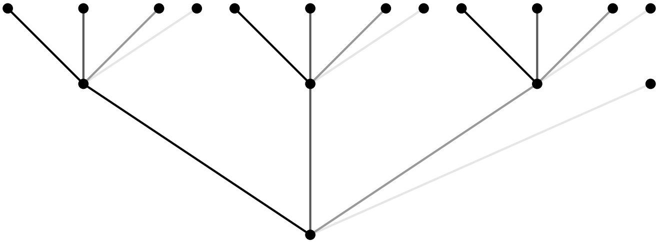

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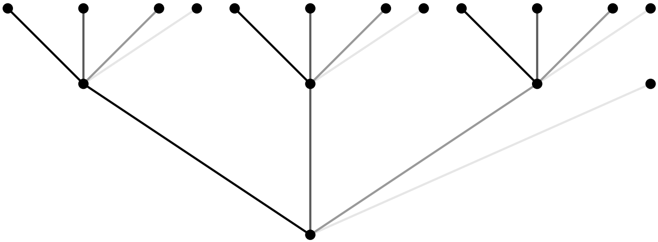

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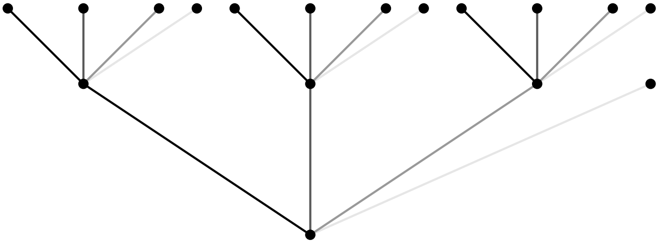

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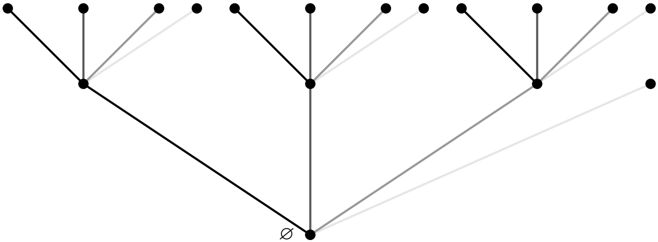

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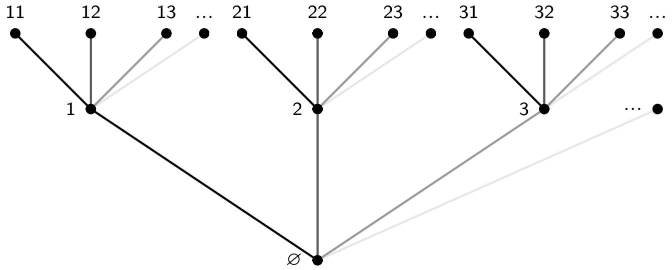

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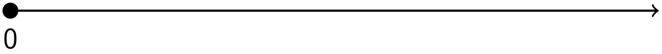

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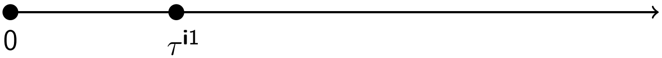

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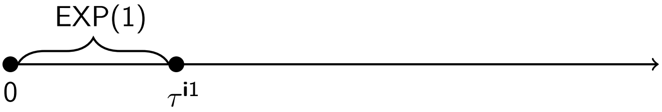

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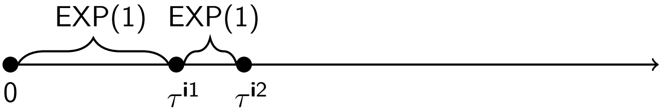

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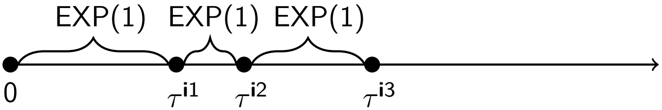

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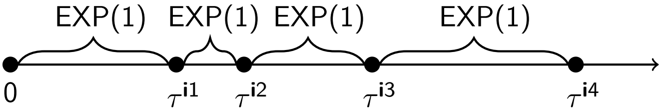

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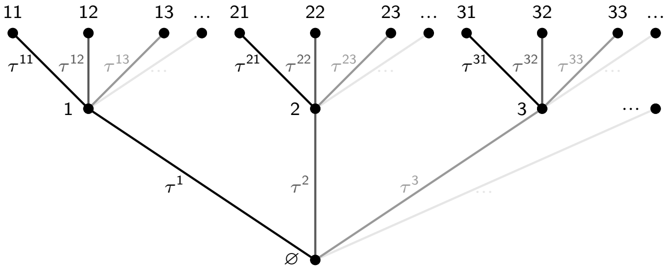

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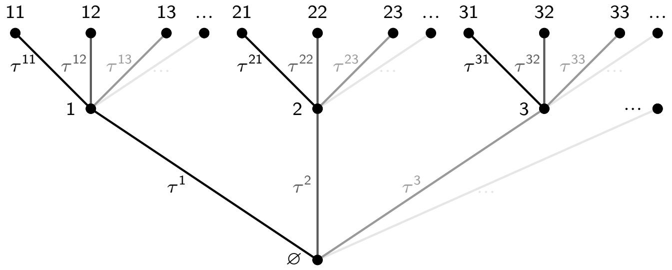

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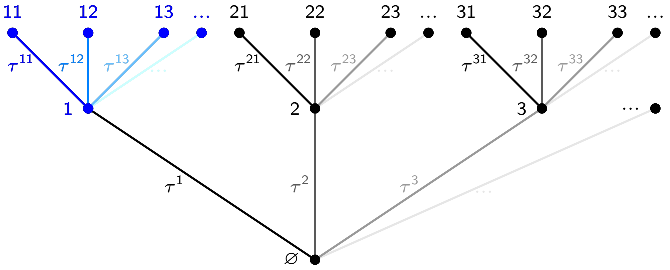

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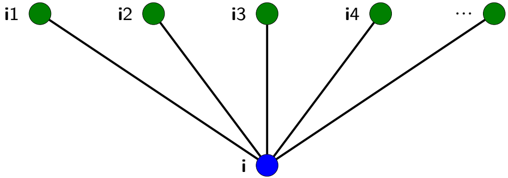

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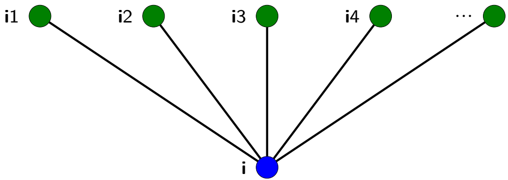

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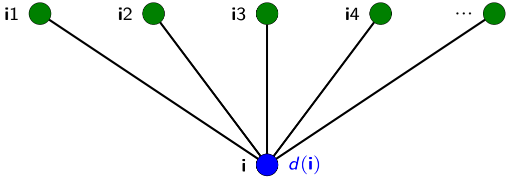

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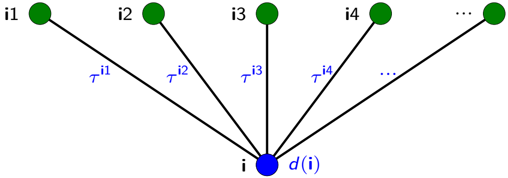

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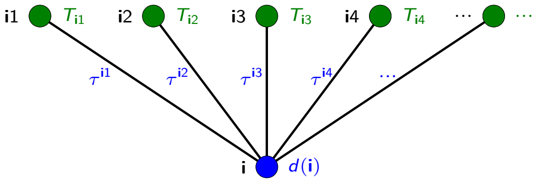

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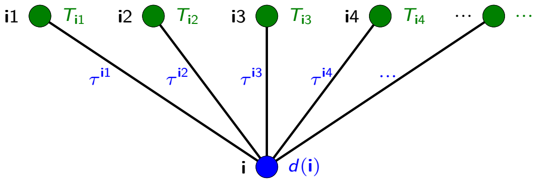

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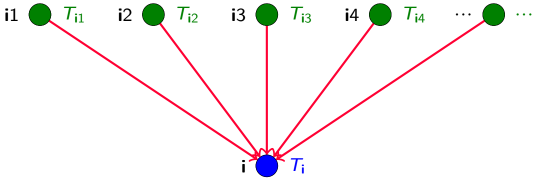

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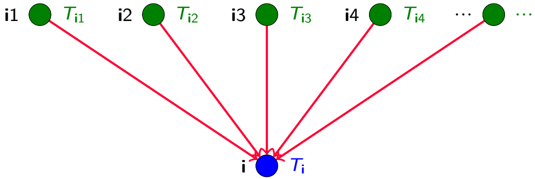

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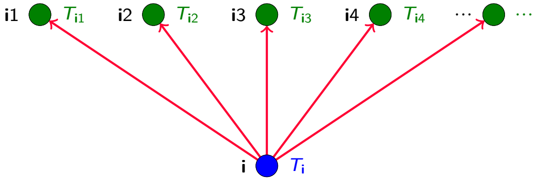

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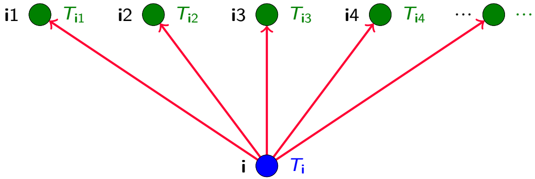

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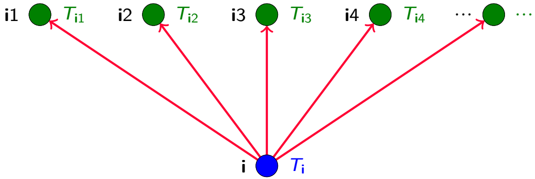

↳ Explicit fixed point equation for the distribution of $T_{\mathbf{i}}$

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Phantom saturation times of the RDCP on the PWIT

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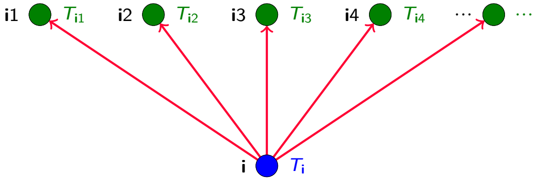

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There exists a critical time $t_c = t_c(\underline{p})$ when a giant connected component emerges in the RDCP on K_n.

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  Namely: t_c can be defined as the first time when the local weak limit has infinitely many vertices with positive probability

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- For $d = 3, 4, \ldots, d_0 - 1$ it is still open

Thank you for your attention!

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https://arxiv.org/pdf/2409.11747