The Brownian Separable Permuton: Universality and Geometric Construction

Márton Szőke

Department of Stochastics, Budapest University of Technology and Economics,
HUN-REN Alfréd Rényi Institute of Mathematics

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Permuton meeting,
Rényi Institute

Recap

Definition (Separable permutation)

A permutation is ***separable*** *if it can be obtained from the permutation $1 \in S_1$ by a sequence of $\oplus$ and $\ominus$.*

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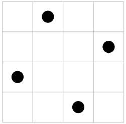

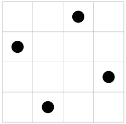

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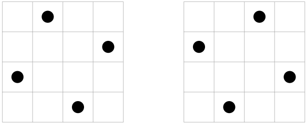
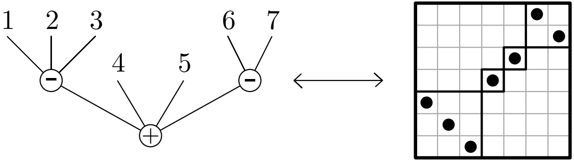


- Bijection with **signed trees** where the signs alternate (level to level)



$= 3214576$

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Theorem (Limit of separable permutations (Bassino et al., 2018))

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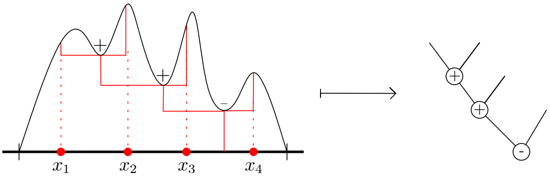
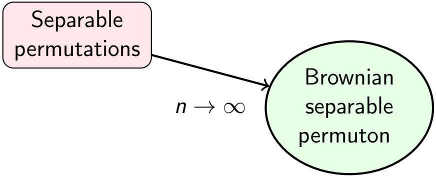
- $(\widetilde{\mathrm{occ}}(\pi, \sigma_n) : \pi \text{ permutation}) \Rightarrow (\Lambda_\pi : \pi \text{ permutation})$
- Λ_π are obtained by a sample of a **signed Brownian excursion** whose strict local minima are decorated with uniform i.i.d. signs in $\{+, -\}$



Motivation

Two natural questions arise:



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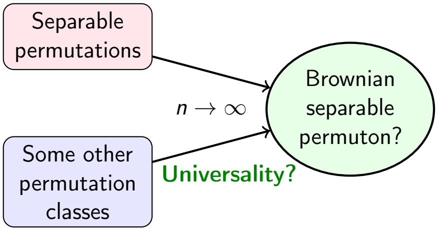

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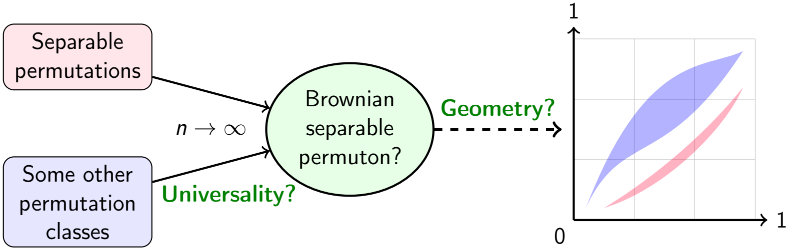
2. **Geometrically**, what does this random measure look like on the unit square? (Maazoun, 2020)



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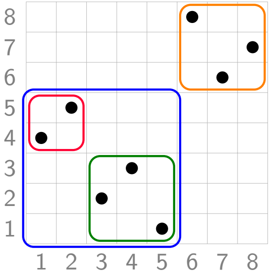
Example: $\pi = 4\,5\,2\,3\,1\,8\,6\,7$



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The ***substitution*** *of permutations* $\alpha_1, \dots, \alpha_m$ *into a permutation* σ *of length m, denoted* $\sigma[\alpha_1, \dots, \alpha_m]$*, is obtained by replacing the i-th element of* σ *with a block isomorphic to* α_i*.*

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Example: $12[\alpha_1, \alpha_2] = \alpha_1 \oplus \alpha_2$, $21[\alpha_1, \alpha_2] = \alpha_1 \ominus \alpha_2$

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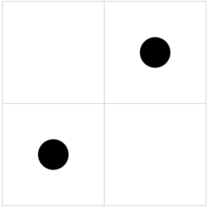

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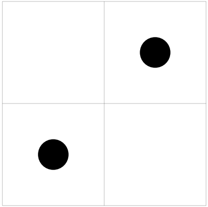

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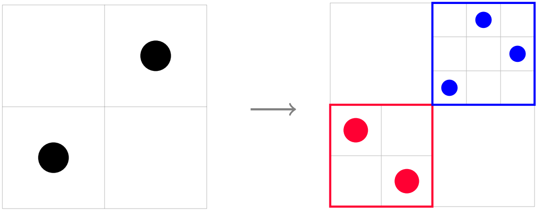

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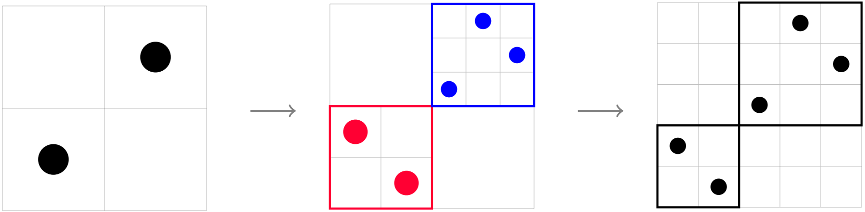

$\sigma = 12$ $\qquad$ $\alpha_1 = 21$, $\alpha_2 = 132$ $\qquad$ $12[21, 132] = 21354$

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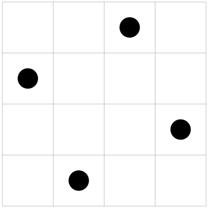

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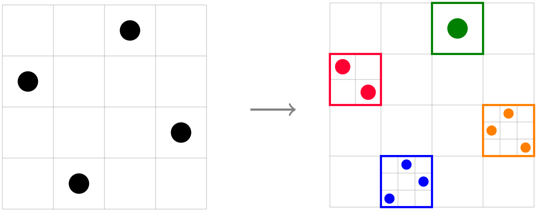

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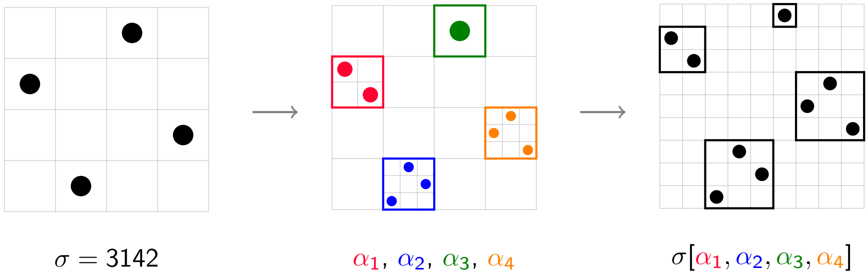

Substitution-closed classes and decomposition

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*A permutation class $\mathcal{C}$ is **substitution-closed** if any substitution of elements from the class remains in the class:*

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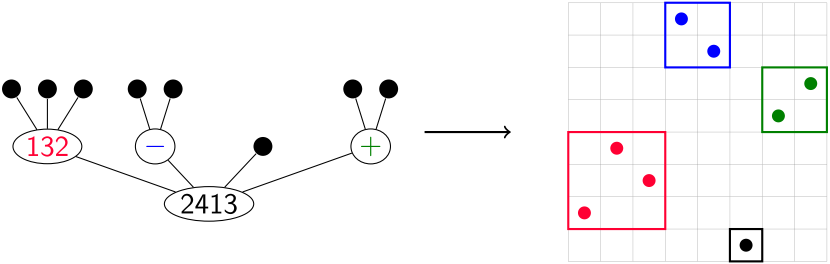

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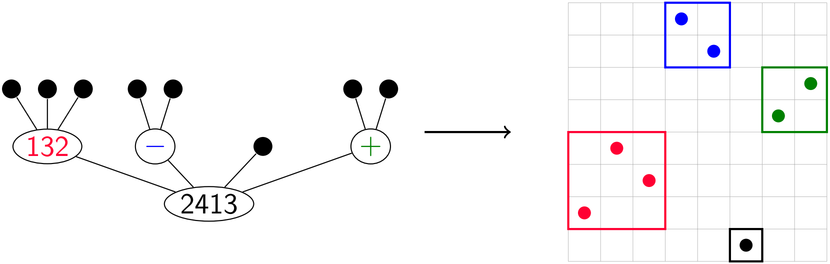

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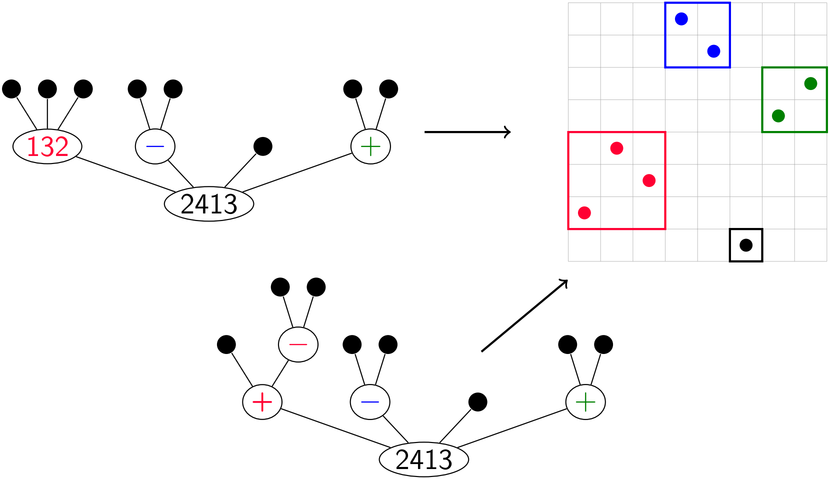

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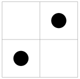

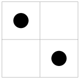

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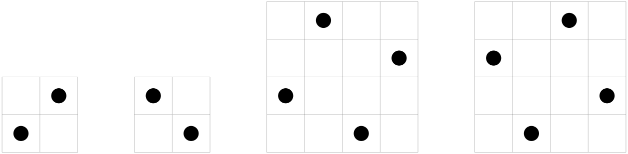

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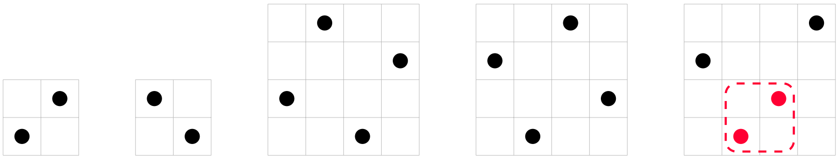

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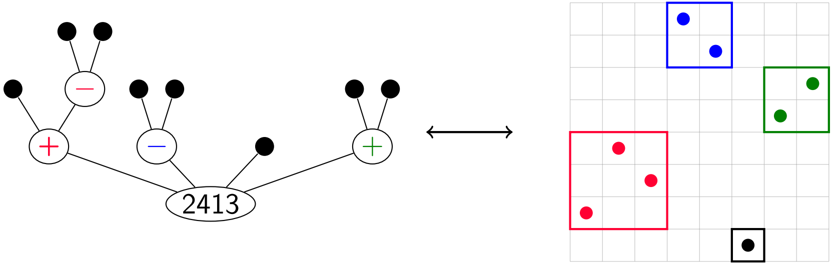

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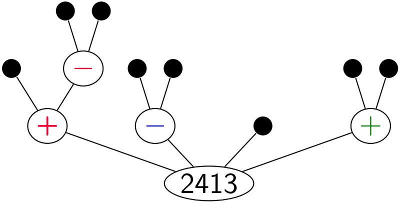

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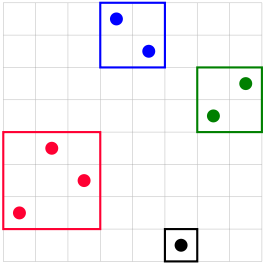

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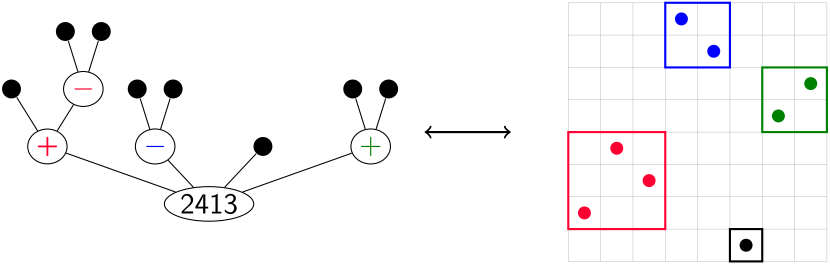

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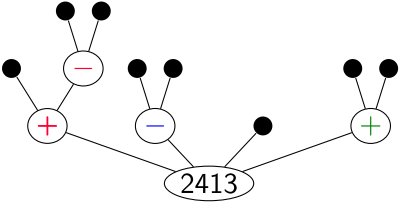

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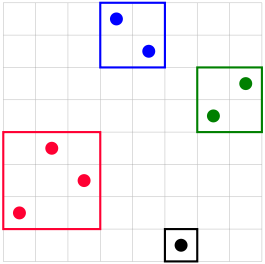

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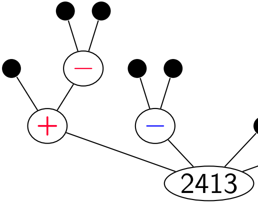

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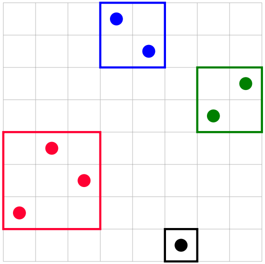

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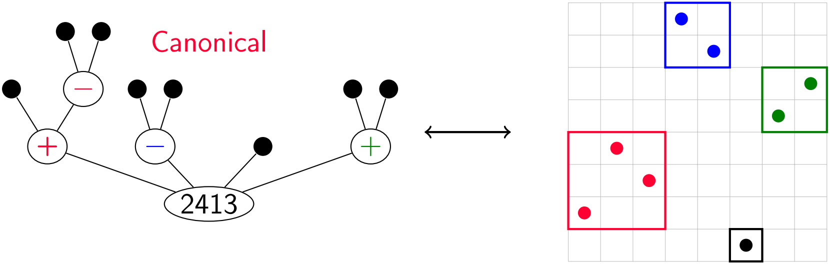

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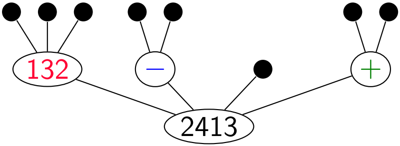
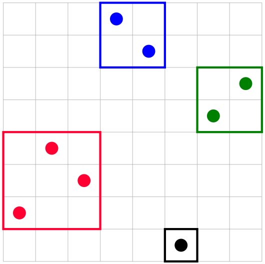


$\longrightarrow$



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Representation of substitution-closed classes

Proposition (Substitution-closed classes as trees)

- *$\mathcal{C}$: a substitution-closed permutation class such that* $12, 21 \in \mathcal{C}$
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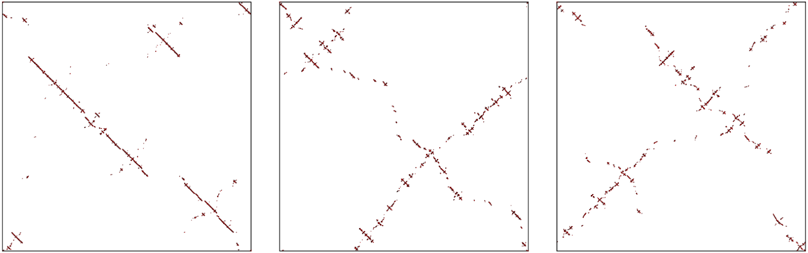

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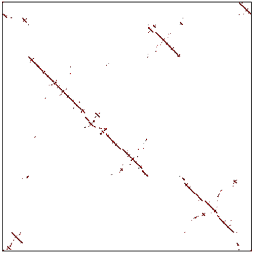

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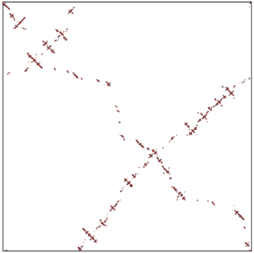

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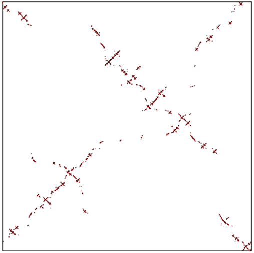

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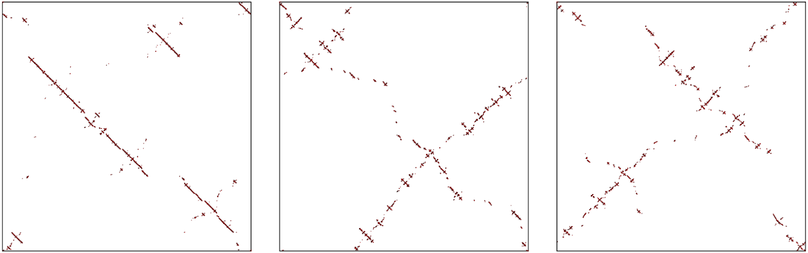

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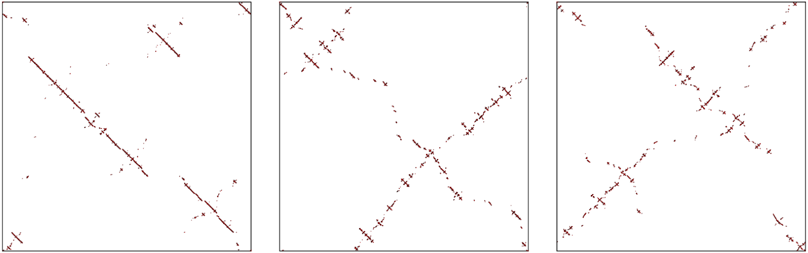

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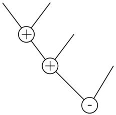

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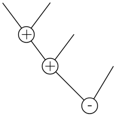

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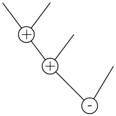

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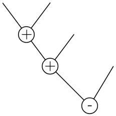

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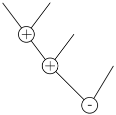

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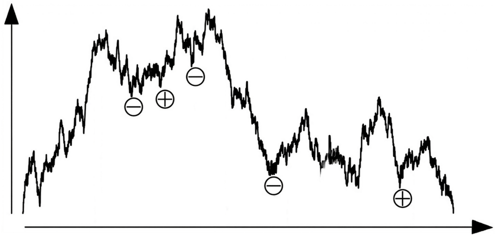
- Can be constructed from a **biased signed Brownian excursion**



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- Counterexample: Av(2413)

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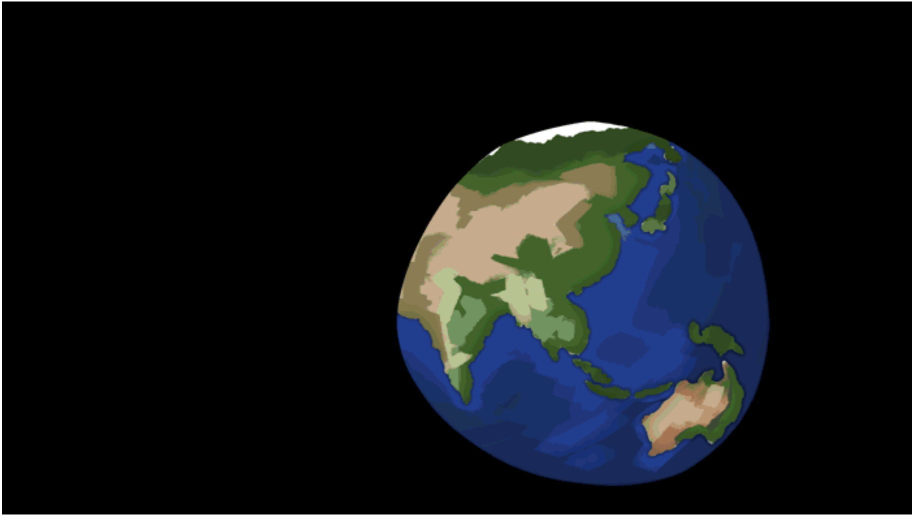

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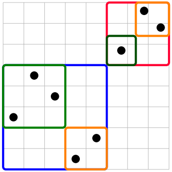

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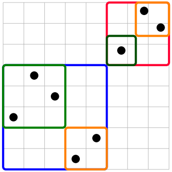

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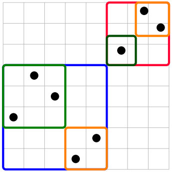

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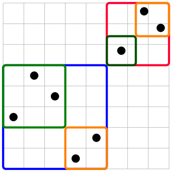

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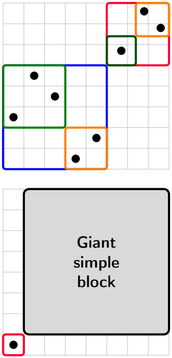

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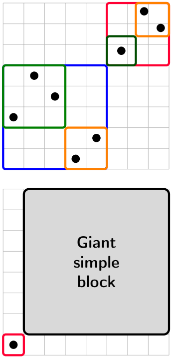

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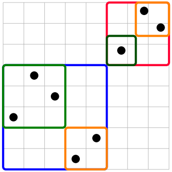

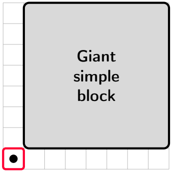

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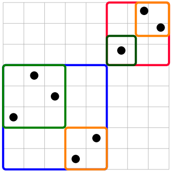

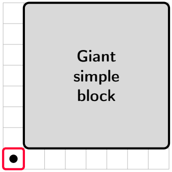

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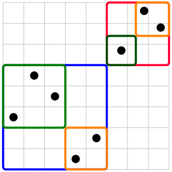

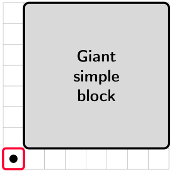

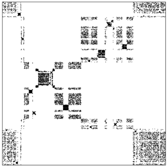

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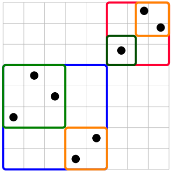

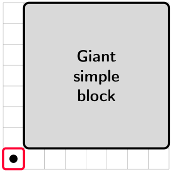

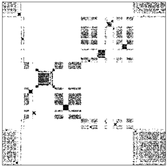

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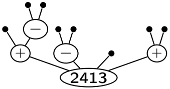

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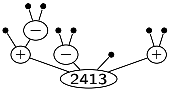

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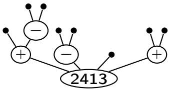

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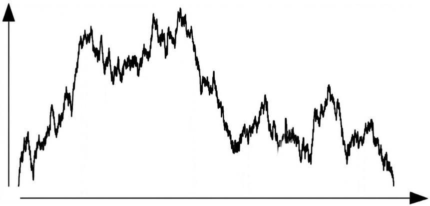

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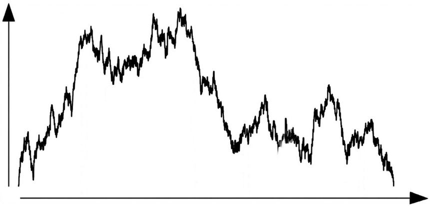

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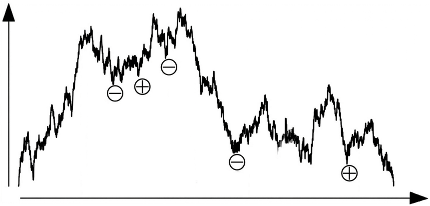

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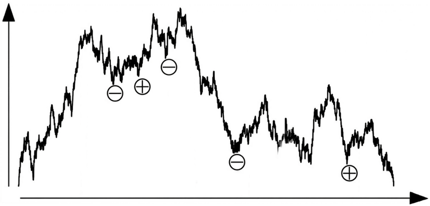

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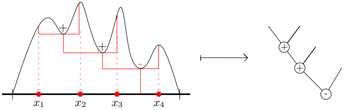

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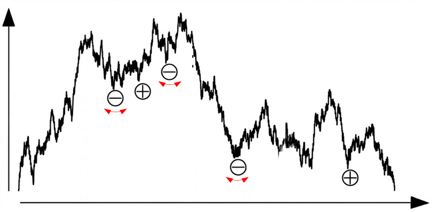

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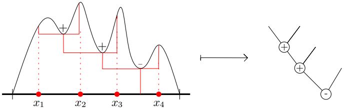
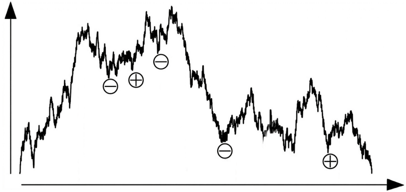


- The signs will dictate where to '**shuffle**' the branches of the tree

The partial order $\triangleleft_e^S$ on the points of $[0, 1]$



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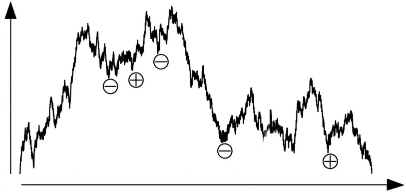

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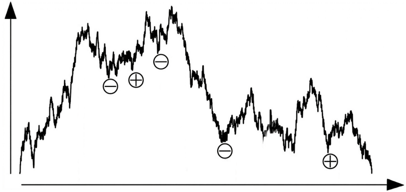

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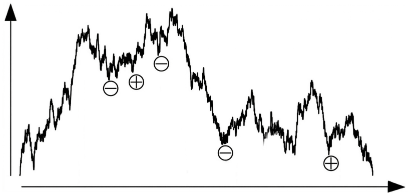

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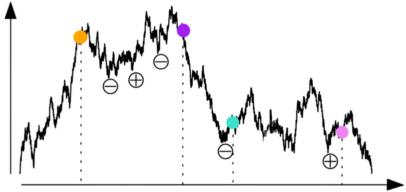

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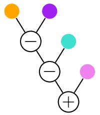

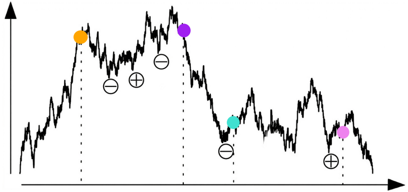

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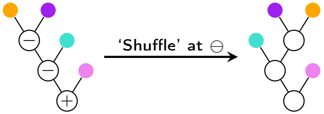

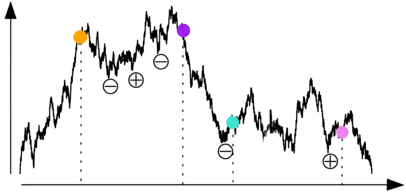

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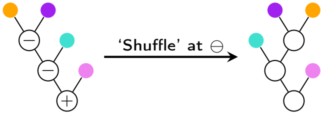

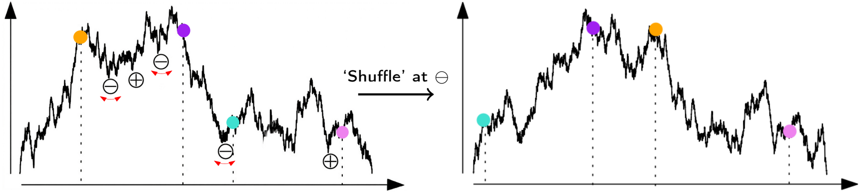

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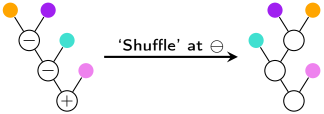

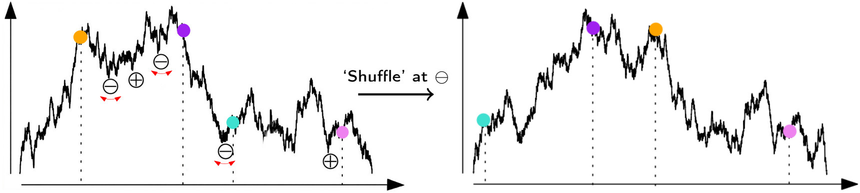

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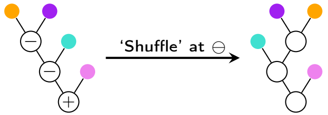
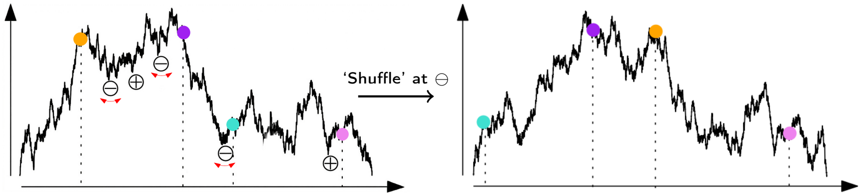
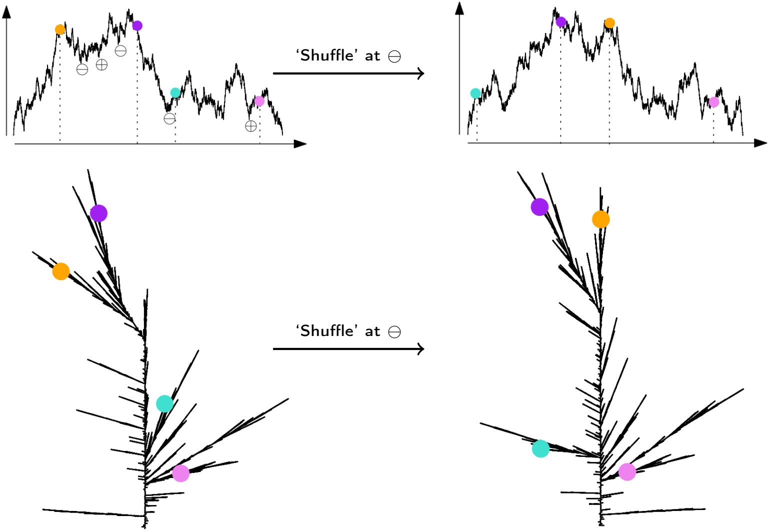




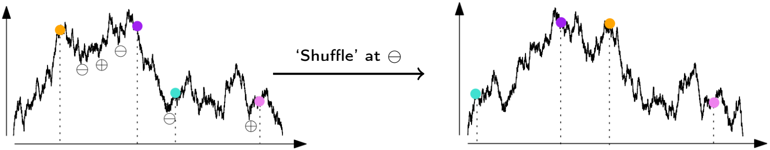
The relation $\lhd_e^S$ is uniquely defined for almost every pair of points
$\Rightarrow$ 'shuffled' order of $[0,1]$

Shuffling of the Brownian CRT



The transformation function

Now let's 'count' how many points precede a given point t in this new, shuffled order



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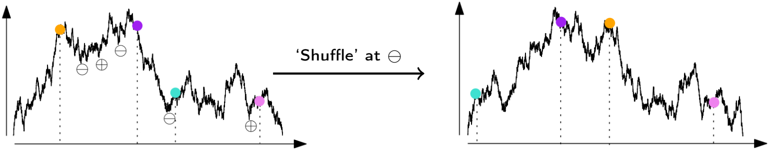

Definition (The transformation function)

Let $\varphi_{e,S} : [0,1] \to [0,1]$ *be defined as:*

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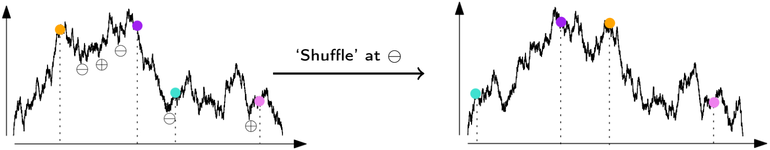

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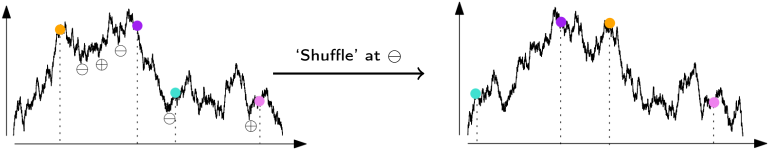

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- The Lebesgue measure (Leb) gives the 'mass' of points preceding t
- This function provides the direct link between the continuous excursion and the geometry of the permuton

The construction: from excursion to permuton

Theorem (Construction of $\mu^{(p)}$ (Maazoun, 2020))

The p-biased Brownian separable permuton ($\mu^{(p)}$) is almost surely the pushforward of the Lebesgue measure via the function $\varphi_{e,S}$:

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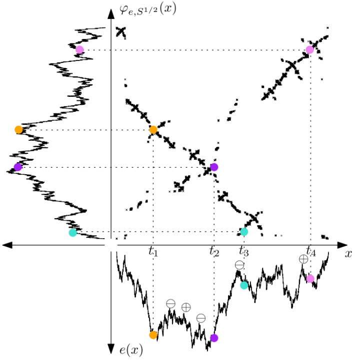
Geometric meaning:

We take a uniform distribution of mass along the x-axis, and lift each particle at position x to the height $y = \varphi_{e,S}(x)$



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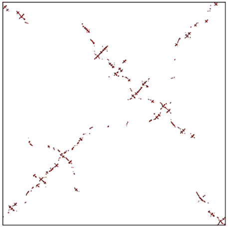
Question: geometrically, what does this random measure look like?



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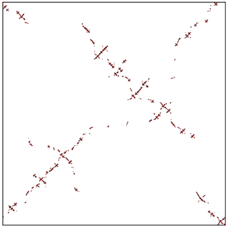

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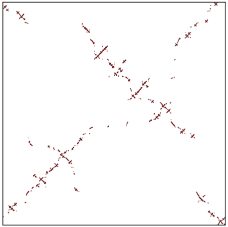

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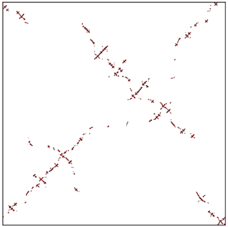

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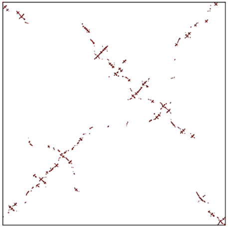

Intuition: due to the infinitely dense local minima of the Brownian excursion, random shuffling happens at every microscopic scale, creating this infinitely fractured dust

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$\mu^{(p)}$ inherits the self-similarity of the Brownian excursion

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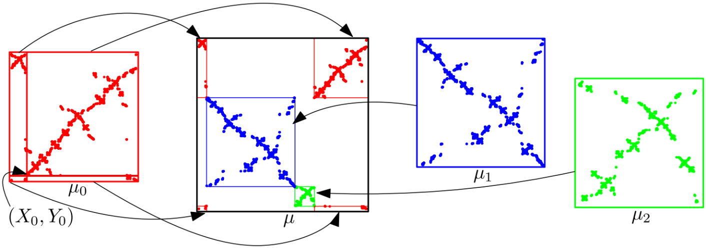

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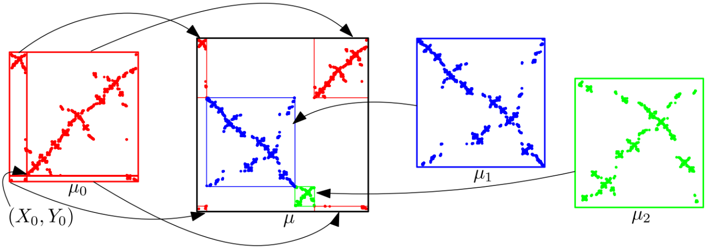

Intuition: deleting a random branching point from the underlying Brownian CRT splits the tree into exactly 3 components

The averaged permuton

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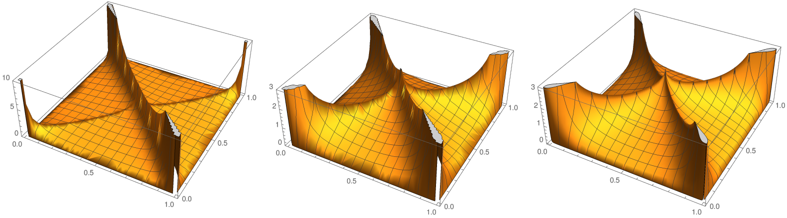
- Taking the expectation of the permuton ($\mathbb{E}[\mu^{(p)}]$) yields a nice, continuous 2D density function:



$p = 0.3$ $p = 0.45$ $p = 0.5$

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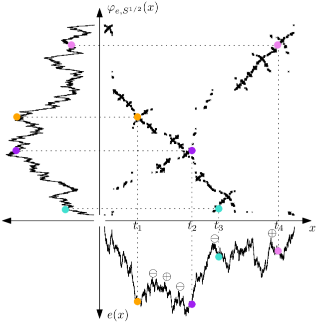

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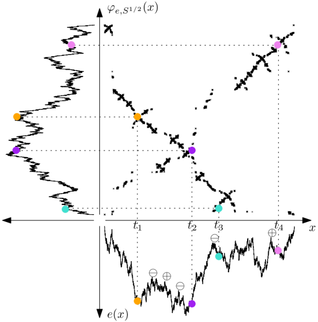

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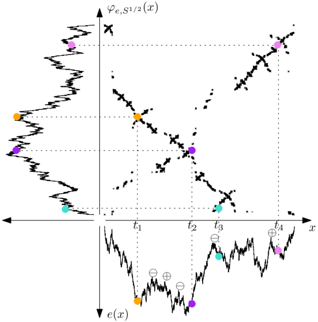

Thank you for your attention!