First midterm

Partial differential equations

Solutions

1. **Find the solution $u \in C^2(\mathbb{R}^2)$ of the following Cauchy problem!**

$$\begin{cases} \partial_{xy} u(x,y) = 0, \\ u(x,0) = x^2, \\ u(0,y) = 4y^2. \end{cases}$$

Solution: By integrating the equation:

$$\partial_{xy} u(x,y) = 0$$

$$\partial_y u(x,y) = f(y)$$

$$u(x,y) = F(y) + g(x)$$

Now by using the boundary conditions:

$$u(x,0) = F(0) + g(x) = x^2$$

$$u(0,y) = F(y) + g(0) = 4y^2$$

Now if $F(0) = c$, then $g(x) = x^2 - c$, and by substituting this into the second equation we get

$$F(y) - c = 4y^2$$

meaning that $F(y) = 4y^2 + c$. Thus, $F(y) = 4y^2 + c$, $g(x) = x^2 - c$ and

$$u(x,y) = 4y^2 + c + x^2 - c = 4y^2 + x^2.$$

2. **Give two, at least second order, two-variable polynomials P and Q in a way that $\Delta P = 0$, $\Delta Q = 0$ and $\Delta(PQ) = 0$.**

Solution: To minimize the amount of calculations, just assume that P and Q are both second order polynomials.

Let us define them as

$$P(x,y) = a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6,$$

$$Q(x,y) = b_1x^2 + b_2xy + b_3y^2 + b_4x + b_5y + b_6.$$

Then

$$\Delta(P) = \partial_x^2 P + \partial_y^2 P = 2a_1 + 2a_3 = 0,$$

$$\Delta(Q) = \partial_x^2 Q + \partial_y^2 Q = 2b_1 + 2b_3 = 0.$$

So choose for example $a_4 = a_5 = a_6 = b_4 = b_5 = b_6 = 0$. Then

$$P \cdot Q = (a_1x^2 + a_2xy - a_1y^2)(b_1x^2 + b_2xy - b_1y^2) =$$

$$= a_1b_1x^4 + a_1b_2x^3y - a_1b_1x^2y^2 + a_2b_1x^3y + a_2b_2x^2y^2 - a_2b_1xy^3 - a_1b_1x^2y^2 - a_1b_2xy^3 + a_1b_1y^4 =$$

$$= a_1b_1x^4 + x^3y(a_1b_2 + a_2b_1) + x^2y^2(a_2b_2 - 2a_1b_1) + xy^3(-a_2b_1 - a_1b_2) + a_1b_1y^4$$

Which means that

$$\Delta(PQ) = 12a_1b_1x^2 + 6xy(a_1b_2 + a_2b_1) + 2(a_2b_2 - 2a_1b_1)x^2 + 2(a_2b_2 - 2a_1b_1)y^2 - 6xy(a_1b_2 + a_2b_1) + 12a_1b_1y^2 =$$

$$= 2x^2(4a_1b_1 + a_2b_2) + 2y^2(4a_1b_1 + a_2b_2)$$

which gives the condition $6a_1b_1 + a_2b_2 = 0$. If we choose $a_1 = b_1 = 1$ and $a_2 = 2$, $b_2 = -2$, then it is satisfied. Therefore a good choice is

$$P(x,y) = x^2 + 2xy - y^2$$

$$Q(x,y) = x^2 - 2xy - y^2$$

3. **Give that solution $u \in C^1(\mathbf{R}^2)$ of the first order partial differential equation**

$$y\,\partial_x u(x,y) + x^3 \partial_y u(x,y) = \frac{1}{2}x^3$$

which equals x^4 on the x-axis!

Solution: This is a first order quasi-linear equation. The auxiliary problem is

$$y\,\partial_x v(x,y,u) + x^3 \partial_y v(x,y,u) + \frac{1}{2}x^3 \partial_u v(x,y,u) = 0,$$

and the characteristic system is

$$\begin{cases} x'(t) = y(t), \\ y'(t) = (x(t))^3, \\ \hat{u}'(t) = \frac{1}{2}(x(t))^3. \end{cases}$$

Our task is now to find first integrals of this system. From the second and third equation, it is easy to see that

$$\frac{1}{2}y'(t) - \hat{u}'(t) = \frac{1}{2}(x(t))^3 - \frac{1}{2}(x(t))^3 = 0,$$

So $\varphi_1(x,y,\hat{u}) = \frac{1}{2}y - \hat{u}$ is a first integral. Also,

$$((x(t))^4)' = 4(x(t))^3 x'(t) = 4y'(t)x'(t) = 4y'(t)y(t) = 2((y(t))^2)'$$

So $\varphi_2(x,y,\hat{u}) = x^4 - 2y^2$ is also a first integral, meaning that we have

$$\frac{1}{2}y - u = \Psi(x^4 - 2y^2),$$

$$u = \frac{1}{2}y - \Psi(x^4 - 2y^2).$$

The boundary condition means that $u(x{,}0) = x^4$. Substituting this into our solution:

$$u(x{,}0) = -\Psi(x^4) = x^4,$$

meaning that $\Psi(x) = -x$, so our solutions are in the form

$$u(x,y) = \frac{1}{2}y + x^4 - 2y^2$$

4. **Define the non-constant two-variable polynomials $a(x,y)$ and $b(x,y)$ in a way that the second order differential operator defined as**

$$Lu = a(x,y)\partial_x^2 u + x\partial_{xy}u + y\partial_{yx}u + b(x,y)\partial_y^2 u$$

is elliptic in the open strip between the lines $x = -y-1$ and $x = -y+1$, but is hyperbolic outside of these lines.

Solution: First let us write the operator into a symmetric form:

$$Lu = a(x,y)\partial_x^2 u + x\partial_{xy}u + y\partial_{yx}u + b(x,y)\partial_y^2 u = a(x,y)\partial_x^2 u + \frac{x+y}{2}\partial_{xy}u + \frac{x+y}{2}\partial_{yx}u + b(x,y)\partial_y^2 u$$

Then the matrix associated with this operator is

$$A = \begin{pmatrix} a(x,y) & \frac{x+y}{2} \\ \frac{x+y}{2} & b(x,y) \end{pmatrix}$$

Then the determinant of this matrix is $\det(A) = a(x,y)b(x,y) - \frac{(x+y)^2}{4}$. Our goal is that this expression is positive if $-y-1 < x < -y+1$, and negative for $-y-1 > x$ and $x > -y+1$. This can be achieved if our expression is a quadratic function, which (for a fixed y value) has roots at $-y-1$ and $-y+1$, and it positive for $x = -y$. Let us then assume that $a(x,y)$ and $b(x,y)$ are both first order polynomials, namely $a(x,y) = a_1x + a_2y + a_3$ and $b(x,y) = b_1x + b_2y + b_3$. Then we want that the expression

$$F(x,y) = (a_1x + a_2y + a_3)(b_1x + b_2y + b_3) - \frac{(x+y)^2}{4}$$

has its roots for $x = -y - 1$ and $x = -y + 1$. Substituting the first one:

$$F(-y-1, y) = (a_1(-y-1) + a_2 y + a_3)(b_1(-y-1) + b_2 y + b_3) - \frac{(-y-1+y)^2}{4} =$$

$$= (y(a_2 - a_1) + a_3 - a_1)(y(b_2 - b_1) + b_3 - b_1) - \frac{1}{4} =$$

$$= y^2(a_2 - a_1)(b_2 - b_1) + y\left[(a_3 - a_1)(b_2 - b_1) + (a_2 - a_1)(b_3 - b_1)\right] + (a_3 - a_1)(b_3 - b_1) - \frac{1}{4}$$

Let us choose

$$a_2 = a_1$$
$$b_2 = b_1$$
$$(a_3 - a_1)(b_3 - b_1) = \frac{1}{4}$$

Also, the equation for the 2nd root is:

$$F(-y+1, y) = (a_1(-y+1) + a_2 y + a_3)(b_1(-y+1) + b_2 y + b_3) - \frac{(-y+1+y)^2}{4} =$$

$$= (y(a_2 - a_1) + a_3 + a_1)(y(b_2 - b_1) + b_3 + b_1) - \frac{1}{4} =$$

$$= y^2(a_2 - a_1)(b_2 - b_1) + y\left[(a_3 + a_1)(b_2 - b_1) + (a_2 - a_1)(b_3 + b_1)\right] + (a_3 + a_1)(b_3 + b_1) - \frac{1}{4}$$

from which we also get that

$$(a_3 + a_1)(b_3 + b_1) = \frac{1}{4}$$

Let us choose $a_3 - a_1 = 1$ and $a_3 + a_1 = \frac{1}{4}$, from which we get that $a_1 = a_2 = -\frac{3}{8}$ and $a_3 = \frac{5}{8}$. Similarly, if $b_3 - b_1 = \frac{1}{4}$ and $b_3 + b_1 = 1$, then $b_1 = b_2 = \frac{3}{8}$ and $b_3 = \frac{5}{8}$. Then $a(x, y) = -\frac{3}{8}x - \frac{3}{8}y + \frac{5}{8}$, $b(x, y) = \frac{3}{8}x + \frac{3}{8}y + \frac{5}{8}$, and

$$F(-y, y) = \frac{5}{8}\frac{5}{8} > 0$$

so this also holds, meaning that this is a good choice.

5. **For an arbitrary, fixed $\phi \in \mathcal{D}(\mathbf{R})$ function let us define $\phi_j(x) := \phi(x + 2020j)$ $(x \in \mathbf{R}, j = 1, 2, \dots)$. Is this sequence convergent in the $\mathcal{D}(\mathbf{R})$ set? If yes, prove it, if not, give a counterexample!**

   Solution: Let us first consider the supports of these functions. It is clear that because of the term $2020j$, the support of the function is always shifted as $j \to \infty$, so there is no such compact set K that $\text{supp}(\phi_j) \subset K$, so it cannot converge.

6. **Let $H \subset \mathbb{R}^2$ be the triangle on the plane with its vertices located at (0,0), (0,1) and (1,0). Let us define the functional $u : \mathcal{D}(H) \to \mathbb{R}$ in the following way:**

$$u(\phi) := \int_0^1 \int_0^x \phi(x, y)\,dy\,dx \qquad (\phi \in \mathcal{D}(H))$$

   Show that $u \in \mathcal{D}'(H)$! Also, prove that $\partial_1 u + \partial_2 u = 0$!

   Solution: It is easy to see that it is indeed linear. For the sequentially continuous property we use the theorem:

$$|u(\phi)| \le \int_0^1 \int_0^x |\phi(x, y)|\,dy\,dx \le \sup_K |\phi| \frac{1}{2}$$

   in which we supposed that $\text{supp}(\phi) \subset K$. (Here instead of $\frac{1}{2}$, one can also write the Lebesgue-measure of K.) So this is indeed a distribution.

   For the second question, let us calculate the two derivatives!

$$\partial_1 u = -\int_0^1 \int_0^x \partial_x \phi(x, y)\,dy\,dx = -\int_0^1 \int_y^1 \partial_x \phi(x, y)\,dx\,dy =$$

$$= -\int_0^1 [\phi(x, y)]_{x=y}^1 dy = -\int_0^1 \phi(1, y) - \phi(y, y)\,dy = \int_0^1 \phi(y, y)\,dy$$

$$\partial_2 u = -\int_0^1 \int_0^x \partial_y \phi(x, y)\,dy\,dx = -\int_0^1 [\phi(x, y)]_{y=0}^x dy = -\int_0^1 \phi(x, x) - \phi(x, 0)\,dy = -\int_0^1 \phi(x, x)\,dx$$

   From which we get that $\partial_1 u + \partial_2 u = 0$.

7. **Let $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x + 1$. Is there such a $u \in \mathcal{D}(\mathbb{R})$ distribution, such that $u' + g = \delta_1$ (in the distribution sense)?**

   Solution: We need a distribution for which $u' = -T_g + \delta_1$. Now we use the theorem stated on the Lecture about the derivative of piece-wise differentiable regular distributions. From this, it is clear that we need a function which has derivative $-g$, and has a jump with height one at $x = 1$. Let us define the following function:

$$f(x) := \begin{cases} -\dfrac{x^2}{2} - x, & \text{if } x < 1, \\ -\dfrac{x^2}{2} - x + 1, & \text{if } x \geq 1, \end{cases}$$

   Then the derivative of T_f is $T_{-g} + \delta_1$, so $u = T_f$.