First midterm

Partial differential equations

Solutions

1. **Solve the following parabolic Cauchy-problem!**

$$\begin{cases} \partial_t u(t,x) - \partial_x^2 u(t,x) = t^2x^3, & \text{on } \mathbb{R}^+ \times \mathbb{R}, \\ u(0,x) = x^2, & (x \in \mathbb{R}). \end{cases}$$

Solution: Let us split this equation into two different ones namely

$$\begin{cases} \partial_t v(t,x) - \partial_x^2 v(t,x) = 0, & \text{on } \mathbb{R}^+ \times \mathbb{R}, \\ v(0,x) = x^2, & (x \in \mathbb{R}), \end{cases}$$

and

$$\begin{cases} \partial_t w(t,x) - \partial_x^2 w(t,x) = t^2x^3, & \text{on } \mathbb{R}^+ \times \mathbb{R}, \\ w(0,x) = 0, & (x \in \mathbb{R}). \end{cases}$$

Using the formula, the solution of the first sub-problem is

$$v(t,x) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-\eta^2}(x - 2\sqrt{t}\eta)^2 d\eta = \frac{1}{\sqrt{\pi}} \left[\int_{\mathbb{R}} e^{-\eta^2}(x^2 - 4x\sqrt{t}\eta + 4t\eta^2)\eta \right] =$$

$$= \frac{1}{\sqrt{\pi}} \left[x^2 \int_{\mathbb{R}} e^{-\eta^2} d\eta - 4x\sqrt{t} \int_{\mathbb{R}} e^{-\eta^2} \eta d\eta + 4t \int_{\mathbb{R}} e^{-\eta^2} \eta^2 d\eta \right]$$

Since $\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-\eta^2} d\eta = 1$, the first term is x^2. The function $e^{-\eta^2}\eta$ is an odd function, so its integral is zero. For the third integral, we do a partial integration:

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\eta^2} \eta^2 d\eta = \frac{1}{\sqrt{\pi}} \left[-\frac{1}{2} e^{-\eta^2} \eta \right]_{-\infty}^{\infty} + \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\eta^2} d\eta = \frac{1}{2}$$

Hee the first term is zero ($e^{-\eta^2}\eta$ tends to zero as $|\eta| \to \infty$), and $\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-\eta^2} d\eta = 1$. Consequently,

$$v(t,x) = x^2 + 2t.$$

For the second part, let us introduce the auxiliary problem

$$\begin{cases} \partial_t \tilde{w}(t,x) - \partial_x^2 \tilde{w}(t,x) = 0, & \text{on } \mathbb{R}^+ \times \mathbb{R}, \\ \tilde{w}(0,x) = \tau^2 x^3, & (x \in \mathbb{R}). \end{cases}$$

The solution of this Cauchy-problem is

$$\tilde{w}(t,x) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-\eta^2} \tau^2 (x - 2\sqrt{t}\eta)^3 d\eta = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-\eta^2} \tau^2 \left(x^3 - 6x^2\sqrt{t}\eta + 12xt\eta^2 - 8t\sqrt{t}\eta^3 \right) d\eta =$$

$$= \frac{1}{\sqrt{\pi}} \left[\tau^2 x^3 \int_{\mathbb{R}} e^{-\eta^2} d\eta - 6x^2\sqrt{t}\tau^2 \int_{\mathbb{R}} e^{-\eta^2} \eta d\eta + 12xt\tau^2 \int_{\mathbb{R}} e^{-\eta^2} \eta^2 d\eta - 8t\sqrt{t}\tau^2 \int_{\mathbb{R}} e^{-\eta^2} \eta^3 d\eta \right] =$$

The second and the fourth functions are odd, so the integrals are zero. Also, $\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-\eta^2} d\eta = 1$ and $\dfrac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\eta^2} \eta^2 d\eta = \frac{1}{2}$, meaning that

$$\tilde{w}(t,x) = \tau^2 x^3 + 6xt\tau^2.$$

Then by the Duhamel principle:

$$w(t,x) = \int_0^t \tilde{w}(t-\tau, x) d\tau = \int_0^t \tau^2 x^3 + 6x(t-\tau)\tau^2 d\tau = \int_0^t \tau^2 x^3 + 6xt\tau^2 - 6x\tau^3 d\tau =$$

$$= (x^3 + 6xt) \left[\frac{\tau^3}{3} \right]_{\tau=0}^{t} - 6x \left[\frac{\tau^4}{4} \right]_{\tau=0}^{t} = (x^3 + 6xt)\frac{t^3}{3} - 6x\frac{t^4}{4} = x^3\frac{t^3}{3} + x\frac{t^4}{2}$$

So the solution is

$$u(x,t) = v(t,x) + w(t,x) = x^2 + 2t + x^3\frac{t^3}{3} + x\frac{t^4}{2}.$$

2. **Let $g, h \in C^1(\mathbb{R})$ monotone increasing functions. Is it true that for the solution u of the hyperbolic equation**

$$\begin{cases} \partial_t^2 u(t,x) - \partial_x^2 u(t,x) = 0, & \textbf{on } \mathbb{R}^+ \times \mathbb{R}, \\ u(0,x) = g(x), & (x \in \mathbb{R}), \\ \partial_t u(0,x) = h(x), & (x \in \mathbb{R}). \end{cases}$$

the function $x \to u(t,x)$ is monotone increasing for any fixed $t > 0$?

Solution: According to the well-known formula,

$$u(t,x) = \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(\xi)d\xi.$$

Then we know that g and h are monotone increasing, meaning that for any $\varepsilon > 0$, $g(x+\varepsilon) \geq g(x)$ and $h(x+\varepsilon) \geq h(x)$. Then

$$u(t, x+\varepsilon) = \frac{1}{2}(g(x+\varepsilon+t) + g(x+\varepsilon-t)) + \frac{1}{2}\int_{x+\varepsilon-t}^{x+\varepsilon+t} h(\xi)d\xi \geq$$

$$\geq \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(\xi+\varepsilon)d\xi \geq$$

$$\geq \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(\xi)d\xi = u(t,x)$$

So this is indeed a monotone increasing function.

3. **Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a sufficiently smooth boundaries, and $p \in C^1(\overline{\Omega})$, for which $p(x) \geq m \geq 0$ for every $x \in \overline{\Omega}$ and $q \in C(\overline{\Omega})$, $q(x) \geq 0$ for every $x \in \overline{\Omega}$ and $q \not\equiv 0$. Let us define the following operator $L : L^2(\Omega) \hookrightarrow L^2(\Omega)$ in the following way:**

$$D(L) := \{u \in C^2(\Omega) \cap C^1(\overline{\Omega}) : u|_{\partial\Omega} + \partial_\nu u|_{\partial\Omega} = 0, Lu \in L^2(\Omega)\}, \qquad Lu := -\mathrm{div}(p\ \mathrm{grad} u) + qu.$$

Show that then this $L : L^2(\Omega) \hookrightarrow L^2(\Omega)$ operator is symmetric, meaning that $\langle Lu, v\rangle_{L^2(\Omega)} = \langle u, Lv\rangle_{L^2(\Omega)}$ for every $u, v \in D(L)$, and L is also strictly positive, i.e. $\langle Lu, u\rangle_{L^2(\Omega)} > 0$ for every $u \in D(L)$, $u \neq 0$.

Solution: First we prove the symmetry property. We are going to use the 2nd Green formula, i.e. $u, v \in C^2(\overline{\Omega}), p \in C^1(\overline{\Omega})$ and Ω is a bounded domain with smooth boundary, then

$$\int_\Omega (v\ \mathrm{div}(p\ \mathrm{grad}(u)) - u\ \mathrm{div}(p\ \mathrm{grad}(v))) = \int_{\partial\Omega} p\ v\ \partial_\nu u - p\ u\ \partial_\nu v\ d\sigma.$$

Using this we get

$$(Lu, v)_{L^2} - (u, Lv)_{L^2} = \int_{\partial\Omega} p\ (v\ \partial_\nu u - u\ \partial_\nu v)\ d\sigma$$

Because of the boundary condition $u|_{\partial\Omega} = -\partial_\nu u$, and $v|_{\partial\Omega} = -\partial_\nu v$. Then

$$(p\ (v\ \partial_\nu u - u\ \partial_\nu v))|_{\partial\Omega} = p\ (-\partial_\nu v\ \partial_\nu u + \partial_\nu u\ \partial_\nu v) = 0$$

So it is symmetric.
Now we prove that L is positive, i.e. $(Lu, u)_{L^2} \geq 0$. We use the first Green formula, i.e.

$$\int_\Omega u\ \mathrm{div}(p\ \mathrm{grad}(u)) = -\int_\Omega p\ \mathrm{grad}(u) \cdot \mathrm{grad}(u) + \int_{\partial\Omega} p\ u\ \partial_\nu u\ d\sigma.$$

Then using this, we get

$$(Lu, u)_{L^2} = \int_\Omega p\ (\mathrm{grad}(u),\ \mathrm{grad}(u)) - \int_{\partial\Omega} p\ u\ \partial_\nu u\ d\sigma + \int_\Omega quu \qquad (1)$$

Then the first and the third terms are non-negative (only zero if $u = 0$), so we only have to consider the second one.
By simple calculations,

$$0 = \int_{\partial\Omega} (u + \partial_\nu u)^2 = \int_{\partial\Omega} (u)^2 + \int_{\partial\Omega} (\partial_\nu u)^2 + 2\int_{\partial\Omega} u\partial_\nu u,$$

from which we get that $\int_{\partial\Omega} u\partial_\nu u \leq 0$, so the second term is also non-negative, meaning that we got positivity.

4. **Let $\Omega \subset \mathbb{R}^n$, $\Omega = B(0,2) \setminus \overline{B(0,1)}$ (where $B(0,r)$ is the ball centered at zero wit radius r). Then for which $\alpha \in \mathbb{R}$ does the following problem have a solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$? Also, give such a solution!**

$$\begin{cases} \Delta u & = \alpha, \qquad (x \in \Omega) \\ \partial_\mu u|_{\partial\Omega} & = 1. \end{cases}$$

Solution: Use the first Green formula with $v \equiv 1$:

$$\int_\Omega 1 \cdot \Delta u = \int_{\partial\Omega} \partial_\mu u d\sigma$$
$$\int_\Omega \alpha = \int_{\partial\Omega} 1 d\sigma$$
$$\alpha(4\pi - \pi) = (4\pi + 2\pi)$$
$$\alpha = 2$$

So the problem has only solution for $\alpha = 2$. We need to show such a solution. By the boundary condition, we need that $\partial_\mu u = \partial_r u = 1$ for $r = 2$ and $\partial_r u = -1$ for $r = 1$. By using the Laplacian in polar coordinates:

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} = 2,$$

which means that we have to solve the Euler equation

$$f''(r) + \frac{1}{r} f'(r) = 2.$$

The solution is $f(r) = \frac{r^2}{2} + c_1 \ln(r) + c_2$, and also $f'(r) = r + \frac{c_1}{r}$. By the boundary condition we need that

$$f'(1) = 1 + c_1 = -1,$$

from which we get $c_1 = -2$, and

$$f'(2) = 2 + \frac{-2}{2} = 2 + (-1) = 1,$$

so this also holds. This means that a sufficient solution is

$$u(x,y) = \frac{x^2 + y^2}{2} - 2\ln(\sqrt{x^2 + y^2}) + c_2,$$

where c_2 is an arbitrary real number.

5. **Let $\Omega = \{x^2 - 2x + y^2 < 3\} \subset \mathbb{R}^2$, and solve the following elliptic boundary-value problem!**

$$\begin{cases} \Delta u & = 6x + 8y, \qquad (x \in \Omega) \\ u|_{\partial\Omega} & = x^3 + y^2. \end{cases}$$

Solution: Let us seek the solution in the form

$$u(x,y) = (x^2 - 2x + y^2 - 3)(ax + by + c) + x^3 + y^2,$$

since in this case the boundary condition is fulfilled. Then

$$\Delta(u(x,y)) = 8ax - 4a + 8by + 4c + 6x + 2.$$

For this to be equal to $6x + 8y$, we need

$$8a + 6 = 6$$
$$8b = 8$$
$$-4a + 4c + 2 = 0$$

From which we get that $a = 0$, $b = 1$ and $c = -\frac{1}{2}$, so our solution is

$$u(x,y) = (x^2 - 2x + y^2 - 3)(y - \frac{1}{2}c) + x^3 + y^2.$$

6. **Solve the following parabolic (mixed) problem!**

$$\begin{cases} \partial_t u(t,x) - \partial_x^2 u(t,x) = \sin(t)\sin(x) & ((t,x) \in \mathbb{R}^+ \times (0,\pi)) \\ u(0,x) = \sin(2x), & (x \in [0,|\pi]) \\ u(t,0) = u(t,\pi) = 0. & (t \in \mathbb{R}_0^+). \end{cases}$$

Solution: Let us split our problem into two, easier sub-problems:

$$\begin{cases} \partial_t u_1(t,x) - \partial_x^2 u_1(t,x) = \sin(t)\sin(x) & ((t,x) \in \mathbb{R}^+ \times (0,\pi)) \\ u_1(0,x) = 0, & (x \in [0,|\pi]) \\ u_1(t,0) = u_1(t,\pi) = 0. & (t \in \mathbb{R}_0^+). \end{cases}$$

and

$$\begin{cases} \partial_t u_2(t,x) - \partial_x^2 u_2(t,x) = 0 & ((t,x) \in \mathbb{R}^+ \times (0,\pi)) \\ u_2(0,x) = \sin(2x), & (x \in [0,|\pi]) \\ u_2(t,0) = u_2(t,\pi) = 0. & (t \in \mathbb{R}_0^+). \end{cases}$$

First we solve the equation for $u_1(t,x)$. Let us search for our solution in the form $u_1(t,x) = c(t)\sin(x)$. Then from the initial value:

$$u_1(0,x) = c(0)\sin(x) = 0,$$

we get that $c(0) = 0$. Also, from the equation:

$$c'(t)\sin(x) + c(t)\sin(x) = \sin(t)\sin(x)$$

$$c'(t) + c(t) = \sin(t)$$

This ordinary diff. equation can be easily solved, and we get

$$c(t) = ce^{-t} + \frac{1}{2}\sin(t) - \frac{1}{2}\cos(t),$$

and because of $c(0) = 0$, $c = \frac{1}{2}$, and

$$c(t) = \frac{1}{2}e^{-t} + \frac{1}{2}\sin(t) - \frac{1}{2}\cos(t),$$

so

$$u_1(t,x) = \left(\frac{1}{2}e^{-t} + \frac{1}{2}\sin(t) - \frac{1}{2}\cos(t)\right)\sin(x).$$

Now we solve the second one. Let us search for our solution in the form $u_2(t,x) = \sum_{k=1}^{\infty} \xi_k(t)\sin(kx)$ (since the eigenfunctions of the laplacian operator are $\sin(kx)$). Then from the initial value:

$$u_2(0,x) = \sum_{k=1}^{\infty} \xi_k(0)\sin(kx) = \sin(2x)$$

which means that $\xi_k \equiv 0$ if $k \neq 2$, and $\xi_2(0) = 1$. Then

$$\xi_2'(t) + 4\xi_2(t) = 0$$

from which we get that (using the initial condition) $\xi_2(t) = e^{-4t}$, and then

$$u_2(t,x) = e^{-4t}\sin(2x).$$

So the solution of the original problem is

$$u(t,x) = u_1(t,x) + u_2(t,x) = \left(\frac{1}{2}e^{-t} + \frac{1}{2}\sin(t) - \frac{1}{2}\cos(t)\right)\sin(x) + e^{-4t}\sin(2x).$$

7. **Let $a > 0$, and then compute the eigenvalues and the eigenvectors of the following operator!**

$$D(L) := \{u \in C^(0,a) \cap C^1([0,a]) : u(0) = 0, u'(a) = 0\}, \qquad Lu := -2u'' + u$$

Solution: The eigenvalue-problem is

$$-2u'' + u = \lambda u$$

$$-u'' = \frac{\lambda - 1}{2}u$$

Depending on the sign of $\frac{\lambda - 1}{2}$, we have three cases:

(a) If $\frac{\lambda-1}{2} > 0$: Then the eigenvectors are in the form

$$u(x) = c_1 \sin\left(\sqrt{\frac{\lambda-1}{2}}x\right) + c_2 \cos\left(\sqrt{\frac{\lambda-1}{2}}x\right)$$

From the boundary conditions we have that $u(0) = c_2 = 0$, and also

$$u'(a) = c_1\sqrt{\frac{\lambda-1}{2}} \cos\left(\sqrt{\frac{\lambda-1}{2}}a\right) = 0$$

If u is not the zero function, then this can only hold if

$$\sqrt{\frac{\lambda-1}{2}}a = \frac{\pi}{2} + k\pi$$

$$\frac{\lambda-1}{2} = \frac{1}{a^2}\left(\frac{\pi}{2} + k\pi\right)^2$$

$$\lambda_k = 1 + \frac{2}{a^2}\left(\frac{\pi}{2} + k\pi\right)^2$$

So the eigenvalues are in this form, and the corresponding eigenfunctions are in the form

$$u_k(x) = c_1 \sin\left(\sqrt{\frac{\lambda_k-1}{2}}x\right).$$

(b) If $\frac{\lambda-1}{2} = 0$: Then the eigenvectors are in the form

$$u(x) = c_1x + c_2$$

From the boundary conditions we have that $u(0) = c_2 = 0$, and also $u'(a) = c_1 = 0$ from which $u \equiv 0$.

(c) If $\frac{\lambda-1}{2} < 0$: Then the eigenvectors are in the form

$$u(x) = c_1 \exp\left(\sqrt{\frac{1-\lambda}{2}}x\right) + c_2 \exp\left(-\sqrt{\frac{1-\lambda}{2}}x\right)$$

From the boundary conditions we have that $u(0) = c_1 + c_2 = 0$, so $c_1 = -c_2$ and also

$$u'(a) = c_1\sqrt{\frac{1-\lambda}{2}} \exp\left(\sqrt{\frac{1-\lambda}{2}}a\right) + c_2\sqrt{\frac{1-\lambda}{2}} \exp\left(-\sqrt{\frac{1-\lambda}{2}}a\right) =$$

$$= c_1\sqrt{\frac{1-\lambda}{2}}\left[\exp\left(\sqrt{\frac{1-\lambda}{2}}a\right) + \exp\left(-\sqrt{\frac{1-\lambda}{2}}a\right)\right] = 0$$

which can only hold if $c_1 = 0$, so $u \equiv 0$.