PDE Mock Midterm 1
2024

1. (Sheet 1, Exercise 3. (a)) Give the solution $u \in C^2(\mathbb{R}^2)$ of the following equation!
$$\begin{cases} \partial_{xy} u = x + y, \\ u(x,x) = x, \\ \partial_x u(x,x) = 0. \end{cases}$$

2. (Sheet 1, Ex. 6.) Suppose that for some $u \in C^4(\mathbb{R}^2)$, we have $\Delta u = 0$. Prove that if $v(x,y) := (x^2 + y^2)u(x,y)$, then $\Delta^2 v = 0$.

3. (Sheet 2, Ex. 1. (e)) Solve the following equation!
$$yz\, \partial_x u(x,y,z) + xz\, \partial_y u(x,y,z) + (x^2 + y^2)\partial_z u(x,y,z) = 0.$$

4. (Sheet 2, Ex. 3. (e)) Solve the following Cauchy problem!
$$\begin{cases} x\partial_x u(x,y) + y\partial_y u(x,y) = u(x,y), \\ u(x,1) = x^2. \end{cases}$$

5. (Sheet 3, Ex. 6.) Give the a, b non-constant polynomials in a way that the differential operator
$$Lu = a(x,y)\partial_x^2 u + x^2 \partial_{xy} u + y^2 \partial_{yx} u + b(x,y)\partial_y^2 u$$
is elliptic inside $B(0,1)$ and inside $\mathbb{R}^2 \setminus \overline{B(0,2)}$, and hyperbolic inside $B(0,2) \setminus \overline{B(0,1)}$.

   Note: The exercise in the midterm will be easier.

6. (Sheet 4, Ex. 1. (c)) Let $\Omega \subset \mathbb{R}^n$ be an open and connected subset, and $\phi \in \mathcal{D}(\Omega)$ (where $\mathcal{D}(\Omega)$ is the set of test functions, as defined on the lecture). Let $\Omega = \mathbb{R}^n$, and $\phi_j(x) := \frac{1}{j}\phi(jx)$ $(x \in \Omega, j \in \mathbb{Z}^+)$. Is it true that this sequence is convergent in the $\mathcal{D}(\Omega)$-sense?

7. (Sheet 4, Ex. 4.) Let $\Omega = (0,2)$ and $u : \mathcal{D}(\Omega) \to \mathbb{R}$ defined as
$$u(\phi) := \sum_{j=1}^{\infty} \phi^{(j)}\left(\frac{1}{j}\right).$$
Show that u is a distribution!