Szám. Mód. Fiz. 1. 2. PZH | -1- | 2018. 12. 3.

1, a, $\text{Rank } A = \dim(\text{Ran } A)$ (képtér dimenziója) 3

b, A $\mathbb{V}$ vektorterében a $\{\underline{v}_i\}_{i\in I} \subset \mathbb{V}$ rendszer generátorrendszer, ha $\forall \underline{a}\in\mathbb{V}$ esetén $\exists\, \alpha_i$ skalárok úgy, hogy véges sok kivétellel $\forall i\in I$-re $\alpha_i = 0$, és $\sum'_{i\in I} \alpha_i \underline{v}_i = \underline{a}$ 3

c, $$\det[a_{i,j}] = \sum'_{\sigma\in S_n} \pi(\sigma)\cdot a_{1,\sigma(1)}\cdot a_{2,\sigma(2)}\cdot\ldots\cdot a_{n,\sigma(n)},$$ ahol

S_n az n-edrendű permutációcsoport, $\pi(\sigma)$ pedig $+1$, ha $\sigma\in S_n$ páros, -1, ha páratlan. 4

2, a, $$\big(A(\alpha f+\beta g)\big)(x) = (\alpha f+\beta g)(2x-1) = \alpha \underbrace{f(2x-1)}_{(Af)(x)} + \beta \underbrace{g(2x-1)}_{(Ag)(x)} =$$

$$= (\alpha Af + \beta Ag)(x) \quad \forall x\text{-re},$$

tehát $A(\alpha f+\beta g) = \alpha Af + \beta Ag$. ✓ 3

b, $h(x) := (Ag)(x) = g(2x-1) = (2x-1)^2 = 4x^2-4x+1$

$$(A^2 g)(x) = (Ah)(x) = h(2x-1) = 4\underbrace{(2x-1)^2}_{4x^2-4x+1} - 4(2x-1)+1 =$$

$$= \underline{\underline{16x^2-24x+9}}$$ 3

c, $y = 2x-1 \iff x = \frac{y+1}{2}$, tehát $\underline{\underline{(A^{-1}f)(x) = f\left(\frac{x+1}{2}\right)}}$ 4

3, a, $\{\underline{a},\underline{b},\underline{c},\underline{d}\}$ lin. függtlen rendszer $\iff \alpha\underline{a}+\beta\underline{b}+\gamma\underline{c}+\delta\underline{d} = \underline{0}$ homogén egyenletrendszernek csak a triviális megoldása van. $\iff$

$\iff [\underline{a}|\underline{b}|\underline{c}|\underline{d}]$ mátrix rangja maximális (4).

Gauss-eliminációval meghatározzuk a vektorokból képzett mátrix rangját. Sor- és oszlop-ekvivalens átalakításokat is végezhetünk.

$$\begin{bmatrix} 1 & -1 & 4 & 1 \\ 2 & 2 & -2 & -1 \\ 0 & 3 & -1 & 1 \\ -1 & 0 & 3 & 2 \end{bmatrix} \begin{matrix} (-2) \\ (+1) \end{matrix} \Leftrightarrow \begin{bmatrix} 1 & -1 & 4 & 1 \\ 0 & 4 & -10 & -3 \\ 0 & 3 & -1 & 1 \\ 0 & -1 & 7 & 3 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & & 4 & 1 \\ 0 & +1 & -7 & -3 \\ 0 & 3 & -1 & 1 \\ 0 & 4 & -10 & -3 \end{bmatrix} \begin{matrix} (-3) \\ (-4) \end{matrix}$$

$$\Leftrightarrow \begin{bmatrix} 1 & -1 & 4 & 1 \\ 0 & 1 & -7 & -3 \\ 0 & 0 & 20 & 10 \\ 0 & 0 & 18 & 9 \end{bmatrix} \begin{matrix} \\ \\ :20 \\ :18 \end{matrix} \Leftrightarrow \begin{bmatrix} 1 & -1 & 4 & 1 \\ 0 & 1 & -7 & -3 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 1 & 1/2 \end{bmatrix} (-1) \Leftrightarrow \begin{bmatrix} \boxed{1} & -1 & 4 & 1 \\ 0 & \boxed{1} & -7 & -3 \\ 0 & 0 & \boxed{1} & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Tehát a rang $= 3 < 4 \Longrightarrow$ nem alkotnak lineárisan független rendszert. 6

b, A generált vektortér 3 dimenziós. (rang) 4

4, $$D = \begin{vmatrix} 3 & +2 & -4 \\ 1 & 0 & -2 \\ 2 & -5 & -6 \end{vmatrix} = \begin{vmatrix} 3 & 2 & 2 \\ 1 & 0 & 0 \\ 2 & -5 & -2 \end{vmatrix} = -1 \cdot \begin{vmatrix} 2 & 2 \\ -5 & -2 \end{vmatrix} = -(-4+10) = \underline{-6}$$

$$D_a = \begin{vmatrix} -7 & 2 & -4 \\ -3 & 0 & -2 \\ -5 & -5 & -6 \end{vmatrix} = -2 \underbrace{\begin{vmatrix} -3 & -2 \\ -5 & -6 \end{vmatrix}}_{18-10=8} + 5 \underbrace{\begin{vmatrix} -7 & -4 \\ -3 & -2 \end{vmatrix}}_{14-12=2} = -16 + 10 = -6 \qquad a = \frac{D_a}{D} = \underline{\underline{1}}$$

$$D_b = \begin{vmatrix} 3 & -7 & -4 \\ 1 & -3 & -2 \\ 2 & -5 & -6 \end{vmatrix} = \begin{vmatrix} 3 & 2 & 2 \\ 1 & 0 & 0 \\ 2 & 1 & -2 \end{vmatrix} = - \underbrace{\begin{vmatrix} 2 & 2 \\ 1 & -2 \end{vmatrix}}_{-4-2} = +6 \; ; \quad b = \frac{D_b}{D} = \underline{\underline{-1}}$$

$$D_c = \begin{vmatrix} 3 & 2 & -7 \\ 1 & 0 & -3 \\ 2 & -5 & -5 \end{vmatrix} = \begin{vmatrix} 3 & 2 & 2 \\ 1 & 0 & 0 \\ 2 & -5 & 1 \end{vmatrix} = - \underbrace{\begin{vmatrix} 2 & 2 \\ -5 & 1 \end{vmatrix}}_{2+10} = -12 \; ; \quad c = \frac{D_c}{D} = \underline{\underline{+2}}$$

5,

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 7 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] (-2) \Longleftrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 2 & -2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \text{csere} \Longleftrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 3 & 2 & -2 & 1 & 0 \end{array}\right] (-3)$$

$$\Longleftrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -2 & 1 & -3 \end{array}\right] (-1) \quad \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 3 & -1 & 3 \\ 0 & 1 & 0 & -2 & 1 & -2 \\ 0 & 0 & 1 & 2 & -1 & 3 \end{array}\right] (-2) \Longleftrightarrow$$

$$\Longleftrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -3 & 7 \\ 0 & 1 & 0 & -2 & 1 & -2 \\ 0 & 0 & 1 & 2 & -1 & 3 \end{array}\right] \quad \text{Tehát } A^{-1} = \begin{bmatrix} 7 & -3 & 7 \\ -2 & 1 & -2 \\ 2 & -1 & 3 \end{bmatrix}$$

(10)

— ∘ —

6,

$$\begin{vmatrix} 1-\lambda & 2 & -1 \\ 0 & -1-\lambda & 0 \\ 0 & 1 & 3-\lambda \end{vmatrix} = -(1+\lambda)(1-\lambda)(3-\lambda)$$

$$\Longrightarrow \lambda_1 = +1, \quad \lambda_2 = -1, \quad \lambda_3 = 3$$

(4)

a, $\lambda_1 = +1$

$$\begin{bmatrix} 0 & 2 & -1 \\ 0 & -2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2y - z \\ -2y \\ y + 2z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \begin{array}{l} x \in \mathbb{R} \\ y = z = 0; \end{array}$$

Egy sajátvektor: $\underline{v}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

(3)

b, $\lambda_2 = -1$

$$\begin{bmatrix} 2 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longleftrightarrow \begin{array}{l} 2x + 2y - z = 0 \\ 0 = 0 \\ y + 4z = 0 \end{array} \Longrightarrow \underline{v}^{(2)} = \begin{bmatrix} -9 \\ 8 \\ -2 \end{bmatrix}$$

(3)

c, $\lambda_3 = 3$

$$\begin{bmatrix} -2 & 2 & -1 \\ 0 & -4 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longleftrightarrow \begin{array}{l} -2x + 2y - z = 0 \\ -4y = 0 \\ y = 0 \end{array} \Longrightarrow \underline{v}^{(3)} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$