MEGOLDÁS: (Számítási Fiz. 1. VD1) — 1 — 2018. 12. 19.

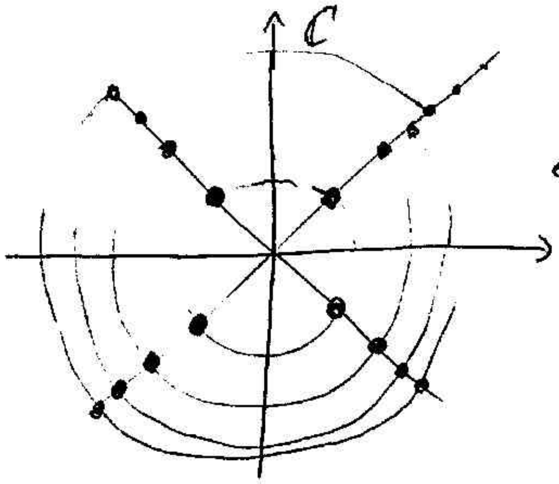
1, $-1 = e^{i\pi}$ ② $\Rightarrow \mathcal{L}n(-1) = \{0 + i\pi(1+2k) \mid k \in \mathbb{Z}\}$ ②

$$z^2 = i\pi(1+2k) = \pi(2l+1)\cdot e^{\pm i\pi/2} \text{ ②}, \quad l \in \mathbb{N}$$

$(k \in \mathbb{Z})$

$$z = \sqrt{\pi(2l+1)}\cdot e^{i\left(n\pi \pm \frac{\pi}{4}\right)} \text{ ②} = \sqrt{\pi(2l+1)}\cdot e^{i\left(n\frac{\pi}{2} + \frac{\pi}{4}\right)} = \sqrt{\pi(2l+1)}\cdot \frac{\pm 1 \pm i}{\sqrt{2}}$$

$l \in \mathbb{N},\ n \in \{0, 1\}$ $\qquad$ $l \in \mathbb{N},\ n \in \{0,1,2,3\}$ $\qquad$ $l \in \mathbb{N}$



rajz: ②

3, a, Legyen $p, q \in P_3$, $\alpha \in \mathbb{C}$. Ekkor

③
$$A(\alpha p + q)(x) = (\alpha p + q)(x-1) = \alpha p(x-1) + q(x-1) = \alpha(Ap)(x) + (Aq)(x) =$$
$$= (\alpha Ap + Aq)(x) \Rightarrow A(\alpha p + q) = \alpha Ap + Aq \ \checkmark$$

b, Legyen $b_n \in P_3$; $b_n(x) = x^n \quad (n = 0, 1, 2, 3)$

$$(Ab_0)(x) = b_0(x-1) = 1 = b_0 \quad ①$$
$$(Ab_1)(x) = b_1(x-1) = x-1 = (b_1 - b_0)(x) \quad ①$$
$$(Ab_2)(x) = b_2(x-1) = (x-1)^2 = x^2 - 2x + 1 = (b_2 - 2b_1 + b_0)(x) \quad ①$$
$$(Ab_3)(x) = b_3(x-1) = (x-1)^3 = x^3 - 3x^2 + 3x - 1 = (b_3 - 3b_2 + 3b_1 - b_0)(x) \quad ②$$

$\boxed{7}$ A kifejtési együtthatók a mátrix oszlopai, tehát:

$$[A] = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ②$$

2, T.: (Dimenziótétel) $\underline{\underline{A}} \in \mathcal{L}(U, W)$ esetén $\dim U = \dim \operatorname{Ker} \underline{\underline{A}} + \dim \operatorname{Ran} \underline{\underline{A}}$

3

$\boxed{7}$ B.: Legyen $\{\underline{b}_1, \underline{b}_2, \dots \underline{b}_k\}$ bázis $\operatorname{Ker} \underline{\underline{A}}$-ban. Ezt egészítsük ki bázissá U-ban: $\{\underline{b}_1, \dots \underline{b}_k; \underline{f}_1, \dots \underline{f}_l\}$ bázis U-ban. Ekkor $\{\underline{\underline{A}}\,\underline{b}_i, \underline{\underline{A}}\,\underline{f}_j\}$ generálja $\operatorname{Ran} \underline{\underline{A}}$-t, de mivel $\underline{\underline{A}}\,\underline{b}_i = \underline{0}$, $\{\underline{\underline{A}}\,\underline{f}_j\}_{j=1}^{l}$ is generálja $\operatorname{Ran} \underline{\underline{A}}$-t. Ugyanakkor $\{\underline{\underline{A}}\,\underline{f}_j\}$ lineárisan független rendszer, hiszen $\sum_{j=1}^{l} \alpha_j \cdot \underline{\underline{A}}\,\underline{f}_j = 0$ esetén $\underline{\underline{A}}\left(\sum_{j=1}^{l} \alpha_j \underline{f}_j\right) = 0$, tehát

$\sum_{j=1}^{l} \alpha_j \underline{f}_j \in \operatorname{Ker} \underline{\underline{A}}$. De $\operatorname{Ker} \underline{\underline{A}}$-t a $\{\underline{b}_i, \underline{f}_j\}$ bázis $\{\underline{b}_i\}$ elemei generálják, tehát $\forall j: \alpha_j = 0$. Így $\dim U = k + l$, $\dim \operatorname{Ker} \underline{\underline{A}} = k$, $\dim \operatorname{Ran} \underline{\underline{A}} = l$, teljesül a dimenziótétel.

— ? —

4, $\boxed{10}$

$$\left[\begin{array}{ccc|c} 3 & -2 & 4 & 4 \\ -2 & 3 & 0 & -1 \\ 1 & -1 & 1 & 1 \\ 2 & 1 & 1 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ -2 & 3 & 0 & -1 \\ 3 & -2 & 4 & 4 \\ 2 & 1 & 1 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & -1 & 3 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -7 & 0 \end{array}\right] \underset{/\times(-1)}{\sim} \left[\begin{array}{ccc|c} \boxed{1} & -1 & 1 & 1 \\ 0 & \boxed{1} & 2 & 1 \\ 0 & 0 & \boxed{1} & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right]$$ 6

$z = 0$

$y + 2z = 1 \Rightarrow y = 1$

$x - y + z = 1$

$x = 1 - 0 + 1 = 2$

3

A mátrix rangja: 3 (vezéregyesek száma) 1

5.* a, $N \in M_n(\mathbb{C})$ nilpotens, ha $\exists k \in \mathbb{N}$ melyre $N^k = 0$ (2)

b, T.: $N \in M_n(\mathbb{C})$ nilpotens $\iff K_N(\lambda) = \lambda^n \cdot (-1)^n$ (karakterisztikus egyenlet monom.) (2)

B.: ($\Rightarrow$) Indirekt. Ha a karakterisztikus egyenletnek van $\mu \neq 0$ gyöke, (3) akkor $\exists \underline{v} \neq \underline{0}$: $\underline{\underline{N}}\,\underline{v} = \mu \underline{v} \Rightarrow \underline{\underline{N}}^l \underline{v} = \mu^l \underline{v} \Rightarrow \underline{\underline{N}}^l \neq 0 \ \forall l$-re ↯ ✓

(3) ($\Leftarrow$) Cayley-Hamilton tétel szerint $K_N(N) = N^n(-1)^n = 0$ ✓

6.* $$\det(A - \lambda I) = \begin{vmatrix} \frac{9}{5} - \lambda & -2/5 & 0 \\ -2/5 & \frac{6}{5} - \lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)\cdot(\lambda^2 - 3\lambda + 2) = -(\lambda-2)^2(\lambda-1)$$

$\lambda_1 = 1, \ \lambda_2 = 2$ (3)

Legyenek a spektrális projekciók P_1 és P_2.

$$\left.\begin{aligned} (1) \quad & P_1 + P_2 = I \\ (2) \quad & 1 \cdot P_1 + 2 \cdot P_2 = A \end{aligned}\right\} \xRightarrow{(2)-(1)} P_2 = A - I = \frac{1}{5}\begin{bmatrix} 4 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \quad (3)$$

$$P_1 = I - P_2 = \frac{1}{5}\begin{bmatrix} 1 & +2 & 0 \\ +2 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (2)$$

$$e^A = e^{\lambda_1} P_1 + e^{\lambda_2} P_2 = \frac{e}{5}\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{e^2}{5}\begin{bmatrix} 4 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} =$$

$$= \frac{1}{5}\begin{bmatrix} e + 4e^2 & 2e - 2e^2 & 0 \\ 2e - 2e^2 & 4e + e^2 & 0 \\ 0 & 0 & 5e^2 \end{bmatrix} \quad (2)$$