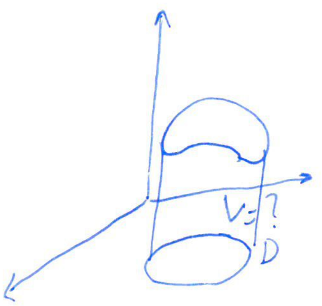
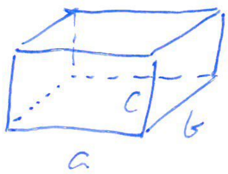
Kettős integrál

Feladat: Legyen $D \subset \mathbb{R}^2$, korlátos tartomány, $f(x,y) \ge 0$ $(x,y) \in D$ folytonos fv. Határozzuk meg az $f(x,y)$ felület és az x,y sík közötti rész térfogatát!

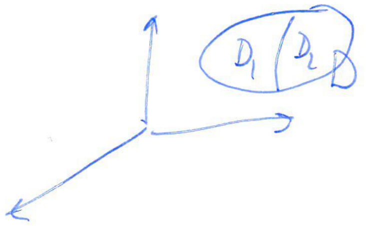


Megoldás: A térfogatnak a következő tulajdonságai vannak:

1, Az a, b, c oldalú téglatest térfogata $a \cdot b \cdot c$

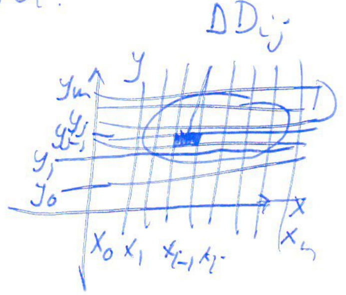


2,



Ha $D = D_1 \cup D_2$, $D_1 \cap D_2 = \emptyset$, akkor $V(D) = V(D_1) + V(D_2)$.

Közelítsük a V térfogatot sok kis alapterületű hasáb térfogatának összegével:



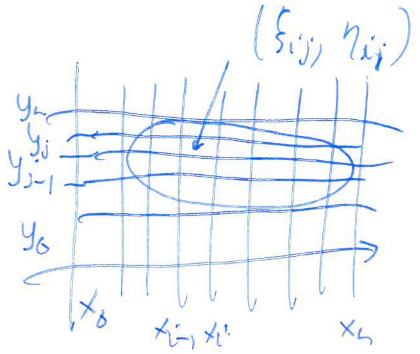
A D tartományt az $x = x_0$, $x = x_1, \dots,$ $x = x_n$ valamint $y = y_0, y = y_1, \dots, y = y_m$ egyenesek segítségével kisebb részekre bontjuk.

$\Delta D_{ij} = \{ (x,y) : x_{i-1} \le x \le x_i,\ y_{j-1} \le y \le y_j \} \cap D$,

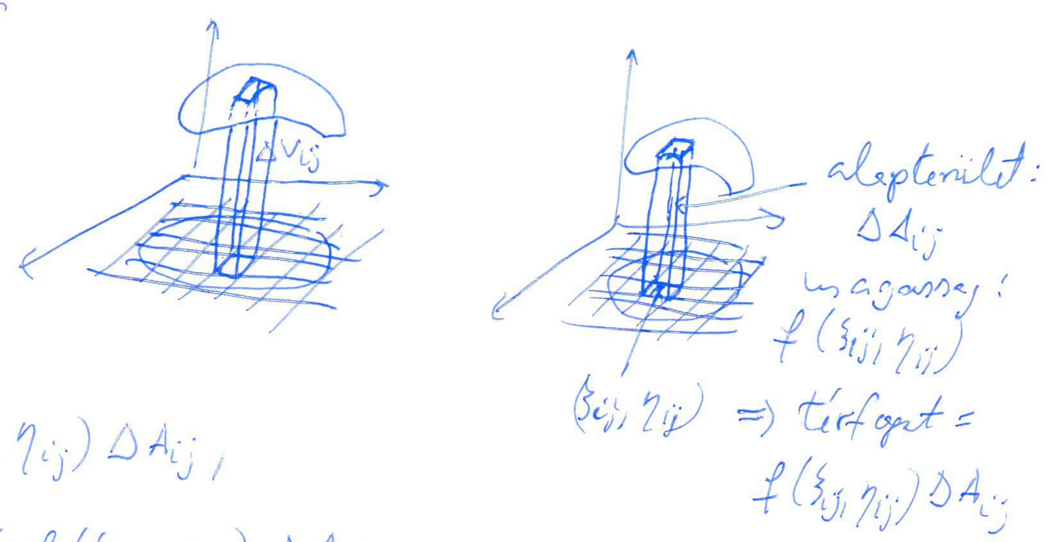
$D = \bigcup_{i,j} \Delta D_{ij}$.

Legyen ΔV_{ij} a ΔD_{ij} feletti térfogat. Ekkor $V = \sum_{i,j} \Delta V_{ij}$

Legyen ΔD_{ij} területe ΔA_{ij}. Ha $\Delta D_{ij} \ne \emptyset$, akkor válasszunk egy pontot ΔD_{ij}-ből: (ξ_{ij}, η_{ij}).



A ΔD_{ij} feletti ΔV_{ij} térfogat jól közelíthető egy ΔA_{ij} alapterületű és $f(\xi_{ij}, \eta_{ij})$ magasságú hasábbal.



Így $\Delta V_{ij} \approx f(\xi_{ij}, \eta_{ij})\Delta A_{ij}$,

$$V = \sum_{i,j} \Delta V_{ij} \approx \sum_{i,j} f(\xi_{ij}, \eta_{ij})\Delta A_{ij}.$$

Def Legyen $f(x,y)$ egy $D \subset \mathbb{R}^2$ korlátos tartományon értelmezett fv. Ha fenti jelölésekkel, továbbá $\Delta x_i = x_i - x_{i-1}$, $\Delta y_j = y_j - y_{j-1}$ esetén létezik a

$$\lim_{\substack{n,m\to\infty \\ \max\{\Delta x_i, \Delta y_j\}\to 0}} \sum_{i,j} f(\xi_{ij}, \eta_{ij})\Delta A_{ij}$$

határérték, akkor ezt $f(x,y)$ kettős integráljának mondjuk.

Jelölés: $\iint_D f(x,y)\,dA$ vagy $\iint_D f(x,y)\,dx\,dy$

Megj 1. Ha $f(x,y) > 0$ folytonos fv, akkor $\iint_D f(x,y)\,dA$ az $f(x,y)$ felület és az xy sík között lévő rész térfogatát számolja ki.

2. Ha $f(x,y)$ folytonos fv D-ben, akkor $\iint_D f(x,y)\Delta A$ az $f(x,y)$ fv előjeles térfogatát számolja ki, azaz az $f(x,y) > 0$ rész térfogatából kivonja az $f(x,y) < 0$ rész térfogatát.

Tétel Ha $f(x,y)$ folytonos a $D \subset \mathbb{R}^2$ korlátos tartományon, akkor létezik az $\iint\limits_D f(x,y)\,dA$ kettős integrál.

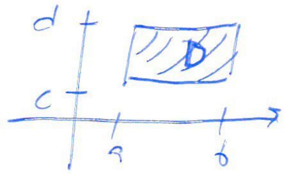
Tulajdonságok

1. $\iint\limits_D c f(x,y)\,dA = c \iint\limits_D f(x,y)\,dA \quad \forall c \in \mathbb{R}$

2. $\iint\limits_D f(x,y) + g(x,y)\,dA = \iint\limits_D f(x,y)\,dA + \iint\limits_D g(x,y)\,dA$

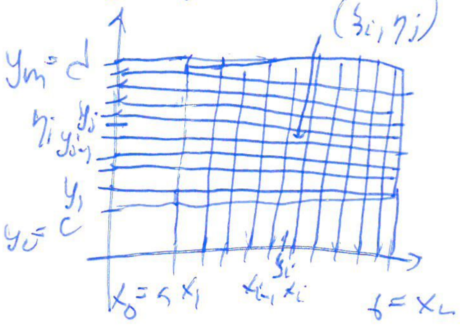
3. Ha $D = D_1 \cup D_2$, $D_1 \cap D_2 = \emptyset$, akkor $\iint\limits_D f(x,y)\,dA = \iint\limits_{D_1} f(x,y)\,dA + \iint\limits_{D_2} f(x,y)\,dA$.

Kettős integrál kiszámítása



1. Téglalap tartományon

$D = \{(x,y):\ a \le x \le b,\ c \le y \le d\}$



$\Delta x_i = x_i - x_{i-1}$

$\Delta y_j = y_j - y_{j-1}$

$$\iint\limits_D f(x,y)\,dA \approx \sum_{j=1}^{m} \sum_{i=1}^{n} f(\xi_i, \eta_j)\,\Delta A_{ij} =$$

$$\sum_{j=1}^{m} \sum_{i=1}^{n} f(\xi_i, \eta_j)\,\Delta x_i\,\Delta y_j =$$

$$\sum_{j=1}^{m} \Delta y_j \underbrace{\sum_{i=1}^{n} f(\xi_i, \eta_j)\,\Delta x_i}_{\approx \int_a^b f(x,\eta_j)\,dx} \approx \sum_{j=1}^{m} \int_a^b f(x,\eta_j)\,dx \cdot \Delta y_j$$

$$\approx \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

Így $\displaystyle \iint\limits_D f(x,y)\,dA = \int_{y=c}^{d} \left(\int_{x=a}^{b} f(x,y)\,dx \right) dy = \int_{x=a}^{b} \left(\int_{y=c}^{d} f(x,y)\,dy \right) dx.$

Pl. $f(x,y) = x + x^2 y$, $D = \{(x,y):\ 0 \le x \le 2,\ 1 \le y \le 3\}$

$$\iint\limits_D f(x,y)\,dA = \int_{x=0}^{2} \Big(\underbrace{\int_{y=1}^{3} x + x^2 y\,dy}_{\left[xy + x^2 \cdot \frac{y^2}{2}\right]_1^{y=3} = 3x + \frac{9}{2}x^2 - \left(x + \frac{1}{2}x^2\right) = 2x + 4x^2} \Big)\,dx = \int_{x=0}^{2} 2x + 4x^2\,dx = \left[x^2 + \frac{4}{3}x^3\right]_0^{x=2} = 4 + \frac{32}{3} = \frac{44}{3}$$

Vagy

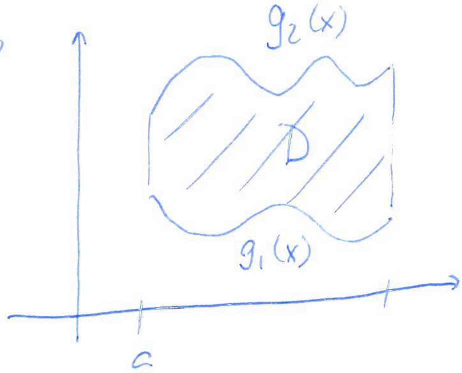
$$\iint_D f(x,y)\,dA = \int_{y=1}^{3}\Big(\underbrace{\int_{x=0}^{2} x + x^2y\,dx}_{\left[\frac{x^2}{2}+\frac{x^3}{3}y\right]_0^{x=2} = \frac{4}{2}+\frac{8}{3}y = 2+\frac{8}{3}y}\Big)dy = \int_{y=1}^{3} 2+\frac{8}{3}y\,dy = \left[2y+\frac{4}{3}y^2\right]_1^{y=3} = 6+12-\left(2+\frac{4}{3}\right) = \frac{44}{3}$$

2. $z = x+y$ sík és a $D = \{(x,y):\ 0 \le x \le 1,\ 0 \le y \le 2\}$ közti rész térfogata?

$$\iint_D x+y\,dA = \int_{x=0}^{1}\Big(\underbrace{\int_{y=0}^{2} x+y\,dy}_{\left[xy+\frac{y^2}{2}\right]_0^{y=2} = 2x+2}\Big)dx = \int_{x=0}^{1} 2x+2\,dx = \left[x^2+2x\right]_0^1 = 3$$

2. Normáltartományok

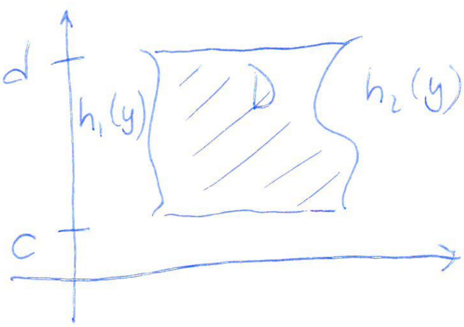
a)



$$D = \{(x,y):\ a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

$$\iint_D f(x,y)\,dA = \int_{x=a}^{b}\Big(\int_{y=g_1(x)}^{g_2(x)} f(x,y)\,dy\Big)dx$$

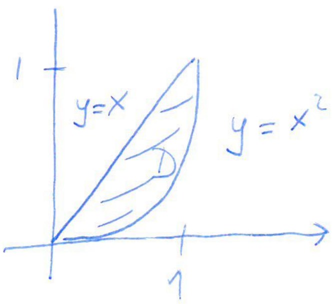
b)



$$D = \{(x,y):\ c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$$

$$\iint_D f(x,y)\,dA = \int_{y=c}^{d}\Big(\int_{x=h_1(y)}^{h_2(y)} f(x,y)\,dx\Big)dy$$

Pl. 1.

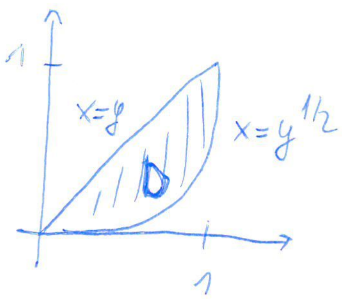


$$\iint_D x+y\,dA = \int_{x=0}^{1}\Big(\underbrace{\int_{y=x^2}^{x} x+y\,dy}_{\left[xy+\frac{y^2}{2}\right]_{x^2}^{y=x} = x^2+\frac{x^2}{2}-\left(x^3+\frac{x^4}{2}\right)}\Big)dx =$$

$$= \int_{x=0}^{1} \frac{3}{2}x^2 - x^3 - \frac{x^4}{2}\,dx = \left[\frac{x^3}{2} - \frac{x^4}{4} - \frac{x^5}{10}\right]_0^1 =$$

$$= \frac{1}{2} - \frac{1}{4} - \frac{1}{10} = \frac{3}{20}$$

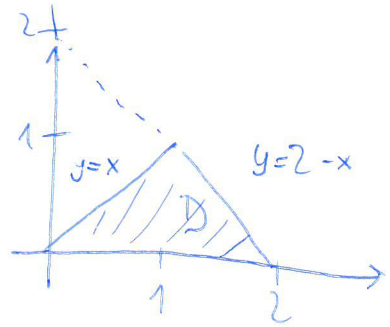
Vagy



$$\iint_D x+y\,dA=\int_{y=0}^{1}\Big(\underbrace{\int_{x=y}^{\sqrt{y}} x+y\,dx}_{\left[\frac{x^2}{2}+xy\right]_{y}^{x=\sqrt{y}}=\frac{y}{2}+y^{3/2}-\left(\frac{y^2}{2}+y^2\right)}\Big)dy=\int_{y=0}^{1}\frac{y}{2}+y^{3/2}-\frac{3}{2}y^2\,dy=$$

$$\left[\frac{y^2}{4}+\frac{2}{5}y^{5/2}-\frac{y^3}{2}\right]_0^1=\frac{1}{4}+\frac{2}{5}-\frac{1}{2}=\frac{3}{20}$$

2.



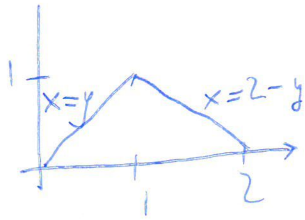
$$\iint_D xy\,dA=?$$

1. Megoldás: $$\iint_D xy\,dA=\int_{x=0}^{1}\Big(\underbrace{\int_{y=0}^{x} xy\,dy}_{\left[x\cdot\frac{y^2}{2}\right]_0^{y=x}=\frac{x^3}{2}}\Big)dx+\int_{x=1}^{2}\Big(\underbrace{\int_{y=0}^{2-x} xy\,dx}_{\left[x\cdot\frac{y^2}{2}\right]_0^{y=2-x}=x\frac{(2-x)^2}{2}=\frac{1}{2}x^3-2x^2+2x}\Big)dx$$

$$=\int_{x=0}^{1}\frac{x^3}{2}\,dx+\int_{x=1}^{2}\frac{1}{2}x^3-2x^2+2x\,dx=$$

$$\left[\frac{x^4}{8}\right]_0^1+\left[\frac{1}{8}x^4-\frac{2}{3}x^3+x^2\right]_1^2=\cancel{\frac{1}{8}}+2-\frac{16}{3}+4-\left(\cancel{\frac{1}{8}}-\frac{2}{3}+1\right)=\frac{1}{3}$$

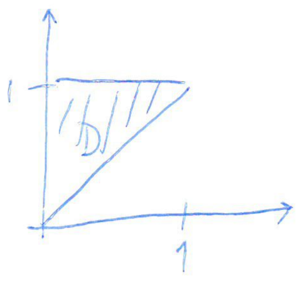
2. Megoldás



$$\iint_D xy\,dA=\int_{y=0}^{1}\Big(\underbrace{\int_{x=y}^{2-y} xy\,dx}_{\left[\frac{x^2}{2}y\right]_{y}^{x=2-y}=\frac{(2-y)^2}{2}y-\frac{y^3}{2}=2y-2y^2}\Big)dy=\int_{y=0}^{1}2y-2y^2\,dy=$$

$$=\left[y^2-\frac{2}{3}y^3\right]_0^{y=1}=1-\frac{2}{3}=\frac{1}{3}$$

3.



$$\iint_D e^{y^2}dA=?$$

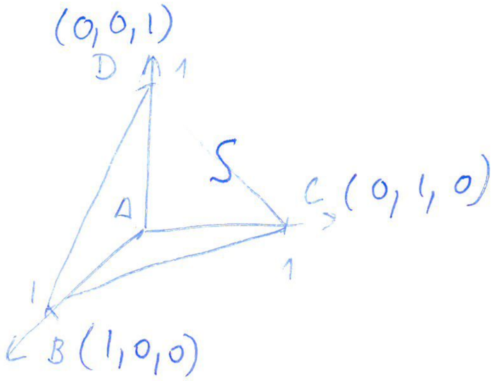
$$\iint_D e^{y^2}dA=\int_{x=0}^{1}\Big(\underbrace{\int_{y=x}^{1} e^{y^2}dy}_{}\Big)dx$$

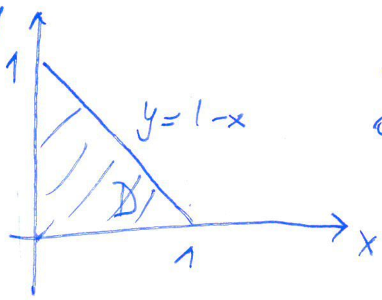
nem tudjuk e^{y^2} primitív fv-ét felírni

$$\iint_D e^{y^2}\,dA = \int_{y=0}^{1} \Big(\underbrace{\int_{x=0}^{y} e^{y^2}\,dx}_{\left[e^{y^2}\cdot x\right]_0^{x=y} = e^{y^2}\cdot y} \Big) dy = \left[\frac{1}{2} e^{y^2}\right]_0^{y=1} = \frac{1}{2}e - \frac{1}{2}$$

Tudjuk: $\frac{d}{dy} e^{y^2} = 2y e^{y^2}$

4. Határozza meg az $A(0,0,0)$, $B(1,0,0)$, $C(0,1,0)$ és $D(0,0,1)$ csúcsú tetraéder térfogatát.





$0 \leq x \leq 1$
$0 \leq y \leq 1-x$

S sík $Ax + By + Cz = 1$ alakú egyenlete:

$B \in S: A\cdot 1 + B\cdot 0 + C\cdot 0 = 1 \Rightarrow A = 1$

$C \in S: A\cdot 0 + B\cdot 1 + C\cdot 0 = 1 \Rightarrow B = 1$

$D \in S: A\cdot 0 + B\cdot 0 + C\cdot 1 = 1 \Rightarrow C = 1$

$x + y + z = 1 \Rightarrow z = 1 - x - y$

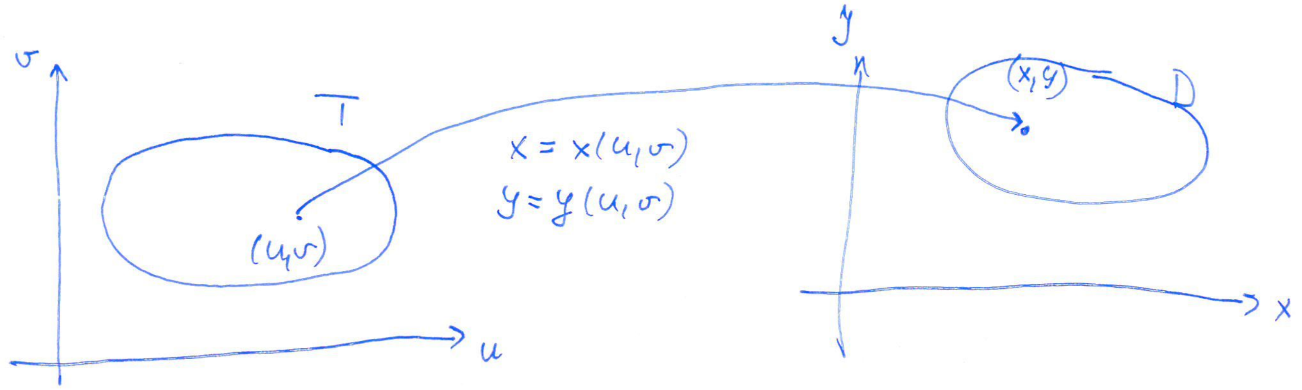
$$V = \int_{x=0}^{1} \Big(\underbrace{\int_{y=0}^{1-x} 1 - x - y\,dy}_{\left[y - xy - \frac{y^2}{2}\right]_0^{y=1-x} = 1 - x - x(1-x) - \frac{(1-x)^2}{2} = \frac{(1-x)^2}{2} = \frac{1}{2} - x + \frac{x^2}{2}} \Big) dx = \int_{x=0}^{1} \frac{1}{2} - x + \frac{x^2}{2}\,dx = \left[\frac{1}{2}x - \frac{x^2}{2} + \frac{x^3}{6}\right]_0^1 =$$

$$= \frac{1}{2} - \frac{1}{2} + \frac{1}{6} = \frac{1}{6}$$

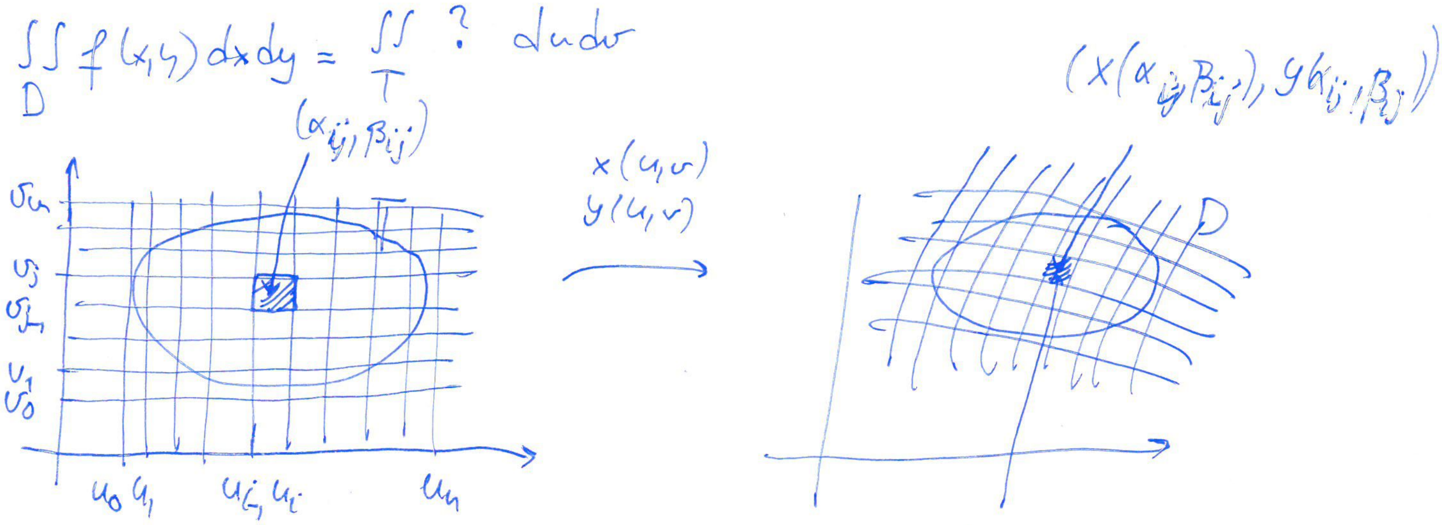
Kettős integrál helyettesítése

Legyen $T, D \subset \mathbb{R}^2$ korlátos tartományok, $f(x,y)$ fv folytonos D-n.

A T tartományt kölcsönösen egyértelműen képezzük le a D tartományra folytonos fvek használva:



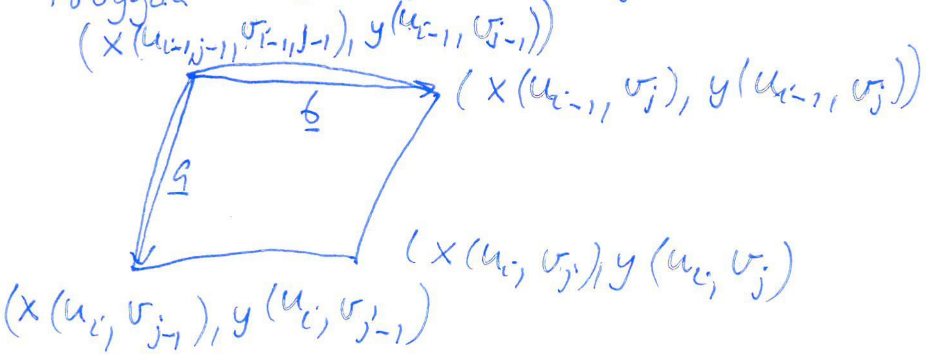
$$\iint_D f(x,y)\,dx\,dy = \iint_T ?\; du\,dv$$



ter $= \Delta A_{ij}$

$$\iint_D f(x,y)\,dx\,dy \approx \sum_{i,j} f\big(x(\alpha_i, \beta_j), y(\alpha_i, \beta_j)\big)\,\Delta A_{ij}$$

Hogyan közelíthetjük a ΔA_{ij} területet?



A ΔA_{ij} területet az $\underline{a}$ és $\underline{b}$ vektorok által meghatározott paralelogramma területével közelíthetjük:

$$\underline{a} = \left(x(u_i, v_{j-1}) - x(u_{i-1}, v_{j-1}),\ y(u_i, v_{j-1}) - y(u_{i-1}, v_{j-1}) \right)$$

$$\underline{b} = \left(x(u_{i-1}, v_j) - x(u_{i-1}, v_{j-1}),\ y(u_{i-1}, v_j) - y(u_{i-1}, v_{j-1}) \right)$$

$$\frac{\partial x}{\partial u}(u_{i-1}, v_{j-1}) = \lim_{u \to u_{i-1}} \frac{x(u, v_{j-1}) - x(u_{i-1}, v_{j-1})}{u - u_{i-1}} \approx \frac{x(u_i, v_{j-1}) - x(u_{i-1}, v_{j-1})}{\underbrace{u_i - u_{i-1}}_{\Delta u_i}}$$

$$\Rightarrow x(u_i, v_{j-1}) - x(u_{i-1}, v_{j-1}) \approx \frac{\partial x}{\partial u}(u_{i-1}, v_{j-1}) \Delta u_i \approx \frac{\partial x}{\partial u}(\alpha_{ij}, \beta_{ij}) \Delta u_i$$

Hasonlóan: $\Delta v_j = v_j - v_{j-1}$

$$y(u_i, v_{j-1}) - y(u_{i-1}, v_{j-1}) \approx \frac{\partial y}{\partial u}(\alpha_{ij}, \beta_{ij}) \Delta u_i$$

$$x(u_{i-1}, v_j) - x(u_{i-1}, v_{j-1}) \approx \frac{\partial x}{\partial v}(\alpha_{ij}, \beta_{ij}) \Delta v_j$$

$$y(u_{i-1}, v_j) - y(u_{i-1}, v_{j-1}) \approx \frac{\partial y}{\partial v}(\alpha_{ij}, \beta_{ij}) \Delta v_j$$

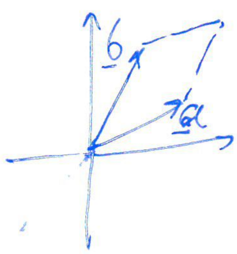
Így

$$\underline{a} \approx \left(\frac{\partial x}{\partial u}(\alpha_{ij}, \beta_{ij}) \Delta u_i,\ \frac{\partial y}{\partial u}(\alpha_{ij}, \beta_{ij}) \Delta u_i \right)$$

$$\underline{b} \approx \left(\frac{\partial x}{\partial v}(\alpha_{ij}, \beta_{ij}) \Delta v_j,\ \frac{\partial y}{\partial v}(\alpha_{ij}, \beta_{ij}) \Delta v_j \right)$$

Az $\underline{a} = (a_1, a_2)$ és $\underline{b} = (b_1, b_2)$ vektorok által meghatározott paralelepipedon területe:

$$ter = \left| \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right|$$



abszolút érték

determináns

Így

$$\Delta A_{ij} \approx |\underline{a} \times \underline{b}| \approx \left| \begin{vmatrix} \frac{\partial x}{\partial u}(\alpha_{ij}, \beta_{ij})\Delta u_i & \frac{\partial y}{\partial u}(\alpha_{ij}, \beta_{ij})\Delta u_i \\ \frac{\partial x}{\partial v}(\alpha_{ij}, \beta_{ij})\Delta v_j & \frac{\partial y}{\partial v}(\alpha_{ij}, \beta_{ij})\Delta v_j \end{vmatrix} \right|$$

$$= \left| \begin{vmatrix} \frac{\partial x}{\partial u}(\alpha_{ij}, \beta_{ij}) & \frac{\partial y}{\partial u}(\alpha_{ij}, \beta_{ij}) \\ \frac{\partial x}{\partial v}(\alpha_{ij}, \beta_{ij}) & \frac{\partial y}{\partial v}(\alpha_{ij}, \beta_{ij}) \end{vmatrix} \right| \Delta u_i \cdot \Delta v_j$$

<u>Def</u> Jacobi-determináns: $J(u_0, v_0) = \begin{vmatrix} \frac{\partial x}{\partial u}(u_0, v_0) & \frac{\partial x}{\partial v}(u_0, v_0) \\ \frac{\partial y}{\partial u}(u_0, v_0) & \frac{\partial y}{\partial v}(u_0, v_0) \end{vmatrix}$

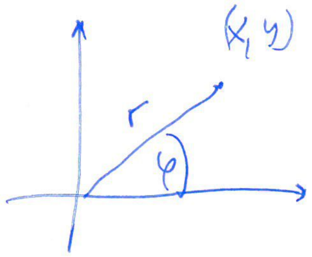
Így $\Delta A_{ij} \approx |J(\alpha_{ij}, \beta_{ij})| \Delta u_i \cdot \Delta v_j$

Tehát $\iint\limits_D f(x,y)\,dA \approx \sum\limits_{i,j} f(x(\alpha_{ij}, \beta_{ij}), y(\alpha_{ij}, \beta_{ij}))|J(\alpha_{ij}, \beta_{ij})| \Delta u_i \cdot \Delta v_j$

$$\approx \iint\limits_T f(x(u,v), y(u,v))|J(u,v)|\,du\,dv.$$

A helyettesítés szabálya: $\iint\limits_D f(x,y)\,dx\,dy = \iint\limits_T f(x(u,v), y(u,v))|J(u,v)|\,du\,dv$

<u>Polár helyettesítés</u>



$x = r\cos\varphi$

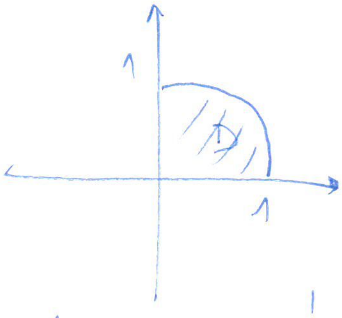
$y = r\sin\varphi$

Jacobi-det: $J(r,\varphi) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial x}{\partial \varphi} \end{vmatrix} =$

$$\begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r\cos^2\varphi - (-r\sin^2\varphi) = r(\cos^2\varphi + \sin^2\varphi) = r$$

Pl. 1. Legyen $D = \{(x,y):\ x^2+y^2 \le 1,\ x, y > 0\}$

$$\iint\limits_D \frac{1}{1+x^2+y^2}\,dx\,dy$$



$x = r\cos\varphi \qquad 0 \le r \le 1$

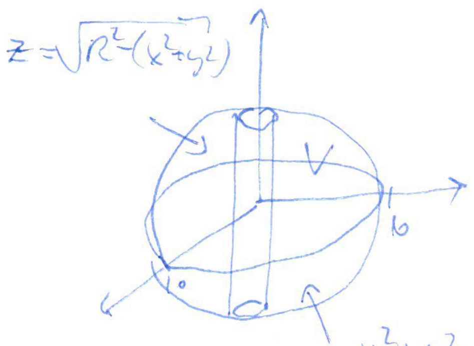
$y = r\sin\varphi \qquad 0 \le \varphi \le \pi/2$

$x^2+y^2 = r^2\cos^2\varphi + r^2\sin^2\varphi = r^2(\cos^2\varphi + \sin^2\varphi) = r^2$

$$\iint_D \frac{1}{1+x^2+y^2}\,dx\,dy = \int_{r=0}^{1} \underbrace{\left(\int_{\varphi=0}^{\pi/2} \frac{1}{1+r^2}\, r\, d\varphi \right)}_{\left[\frac{r}{1+r^2}\varphi\right]_0^{\varphi=\pi/2} = \frac{\pi}{2}\cdot\frac{r}{1+r^2}} dr = \int_{r=0}^{1} \frac{\pi}{2}\cdot\frac{r}{1+r^2}\,dr =$$

$$\int_{r=0}^{1} \frac{\pi}{4}\cdot\frac{2r}{1+r^2}\,dr = \left[\frac{\pi}{4}\ln(1+r^2)\right]_0^{r=1} = \frac{\pi}{4}\ln 2 - \frac{\pi}{4}\ln 1 = \frac{\pi}{4}\ln 1.$$

2. Egy 10 cm sugarú tömör gömböt egy 1 cm átmérőjű fúróval középen átfúrunk. Mekkora a megmaradó rész térfogata?



$$V = V_{gömb} - V_{közép} = \frac{4\cdot 10^3\pi}{3} - \left(\frac{4\pi}{3}\cdot 10^3 - \frac{4\pi}{3}\cdot 99{,}75^{3/2}\right) = \frac{4\pi}{3}\cdot 99{,}75^{3/2}$$

$$V_{közép} = 2\iint_D \sqrt{100-(x^2+y^2)}\,dx\,dy =$$

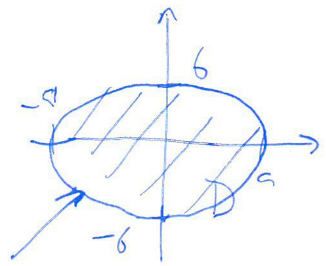
$x = r\cos\varphi \qquad 0 \le r \le 0{,}5$

$y = r\sin\varphi \qquad 0 \le \varphi \le 2\pi$

$$= 2\int_{r=0}^{0,5} \underbrace{\left(\int_{\varphi=0}^{2\pi} \sqrt{100-r^2}\cdot r\, d\varphi \right)}_{\left[\sqrt{100-r^2}\cdot r\cdot\varphi\right]_0^{\varphi=2\pi} = \sqrt{100-r^2}\,2r\pi} dr = 2\int_{r=0}^{0,5} -\pi(100-r^2)^{1/2}(-2r)\,dr =$$

$$2\left[-\pi\,\frac{(100-r^2)^{3/2}}{3/2}\right]_0^{r=0,5} = -\frac{4\pi\cdot 99{,}75^{3/2}}{3} - \left(-\frac{4\pi}{3}\cdot 10^{3/2}\right)$$

1. Határozzuk meg az $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ellipszis területét!



$x = a\cos\varphi$

$y = b\sin\varphi \qquad 0 \le \varphi \le 2\pi$

$$\text{Ter} = \iint_D 1\,dx\,dy = \int_{r=0}^{1}\underbrace{\left(\int_{\varphi=0}^{2\pi} 1\cdot abr\,d\varphi\right)}_{[abr\varphi]_0^{\varphi=2\pi} = 2\pi abr} dr \overset{(*)}{=}$$

D paraméterezése

$x = a r\cos\varphi \qquad 0 \le r \le 1$

$y = b r\sin\varphi \qquad 0 \le \varphi \le 2\pi$

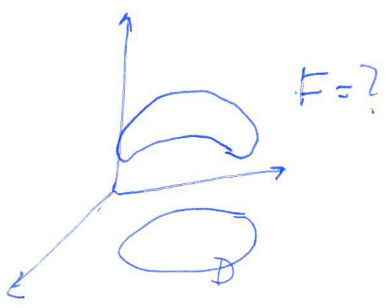
$$J = \left|\begin{matrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi}\end{matrix}\right| = \left|\begin{matrix} a\cos\varphi & -ar\sin\varphi \\ b\sin\varphi & br\cos\varphi\end{matrix}\right| =$$

$abr\cos^2\varphi - (-abr\sin^2\varphi) = abr(\cos^2\varphi + \sin^2\varphi) = abr$

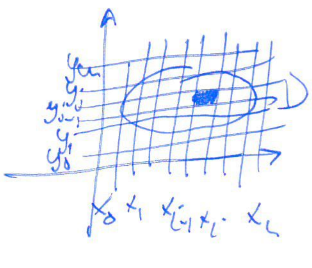
$$\overset{(*)}{=} \int_{r=0}^{1} 2\pi abr\,dr = \left[ab\pi r^2\right]_0^{r=1} = ab\pi$$

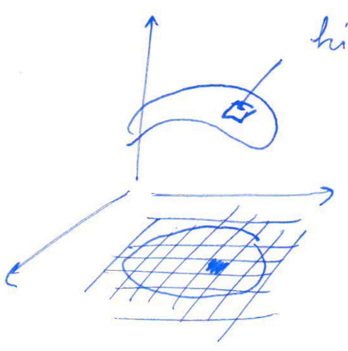
Felszínszámítás

Feladat: Határozzuk meg az $f(x,y)$, $(x,y)\in D$ felület felszínét!



Megoldás: Feltesszük, hogy $f(x,y)$, $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ függvények folytonosak.

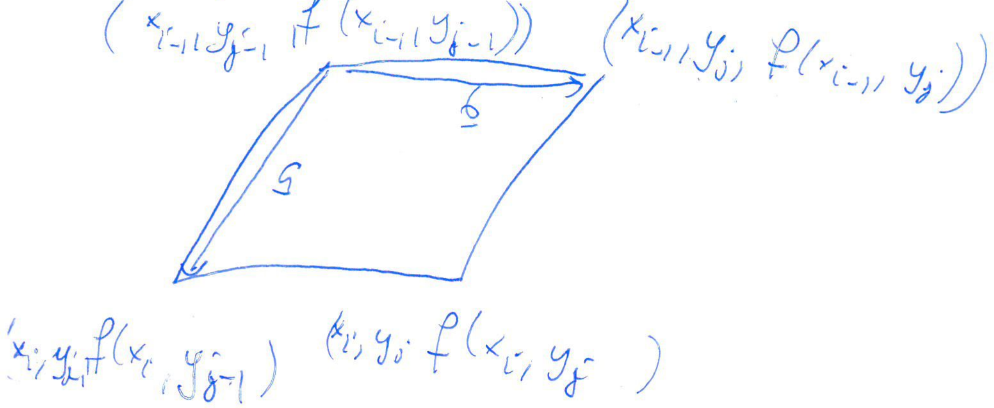




kicsi felületdarab: ΔF_{ij}

$$F = \sum_{i,j} \Delta F_{ij}$$

ΔF_{ij} kinagyítva:



ΔF_{ij} jól közelíthető egy

$$\underline{a} = (0,\ y_j - y_{j-1},\ f(x_{i-1}, y_j) - f(x_{i-1}, y_{j-1}))$$

$$\underline{b} = (x_i - x_{i-1},\ 0,\ f(x_i, y_{j-1}) - f(x_{i-1}, y_{j-1}))$$

oldalú paralelogramma területével.

$$f'_y(x_{i-1}, y_j) = \lim_{y \to y_j} \frac{f(x_{i-1}, y_j) - f(x_{i-1}, y)}{y_j - y} \approx \frac{f(x_{i-1}, y_j) - f(x_{i-1}, y_{j-1})}{\underbrace{y_j - y_{j-1}}_{\Delta y_j}}$$

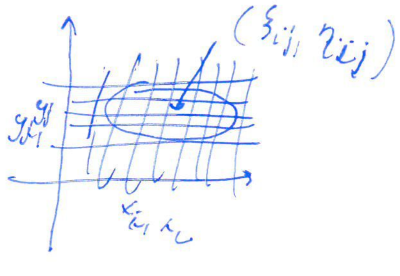
$$\Rightarrow f(x_{i-1}, y_j) - f(x_{i-1}, y_{j-1}) \approx f'_y(x_{i-1}, y_j)\, \Delta y_j$$

$$f'_x(x_i, y_{j-1}) = \lim_{x \to x_i} \frac{f(x_i, y_{j-1}) - f(x, y_{j-1})}{x_i - x} \approx \frac{f(x_i, y_{j-1}) - f(x_{i-1}, y_{j-1})}{\underbrace{x_i - x_{i-1}}_{\Delta x_i}}$$

$$\Rightarrow f(x_i, y_{j-1}) - f(x_{i-1}, y_{j-1}) \approx f'_x(x_i, y_{j-1})\, \Delta x_i$$

Tehát $\underline{a} \approx (0,\ \Delta y_j,\ f'_y(x_{i-1}, y_j)\, \Delta y_j) = (0, 1, f'_y(x_{i-1}, y_j)) \cdot \Delta y_j$

$\underline{b} \approx (\Delta x_i,\ 0,\ f'_x(x_i, y_{j-1})\, \Delta x_i) = (1, 0, f'_x(x_i, y_{j-1})) \cdot \Delta x_i$



$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ folytonos $\Rightarrow$

$$\underline{a} \approx (0, 1, f'_y(\xi_{ij}, \eta_{ij})) \cdot \Delta y_j$$

$$\underline{b} \approx (1, 0, f'_x(\xi_{ij}, \eta_{ij})) \cdot \Delta x_i$$

Az $\underline{a}$ és $\underline{b}$ oldalú paralelogramma területe $|\underline{a}\times\underline{b}|$.

$$|\underline{a}\times\underline{b}| \approx \left| \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & \Delta y_j & f'_y(\xi_{ij},\eta_{ij})\Delta y_j \\ \Delta x_i & 0 & f'_x(\xi_{ij},\eta_{ij})\Delta x_i \end{vmatrix} \right| =$$

abszolút érték

determináns

$$\left| \left(f'_x(\xi_{ij},\eta_{ij})\,\Delta x_i\cdot\Delta y_j,\ f'_y(\xi_{ij},\eta_{ij})\,\Delta x_i\cdot\Delta y_j,\ -\Delta x_i\cdot\Delta y_j \right) \right|$$

$$= \sqrt{\left(f'_x(\xi_{ij},\eta_{ij})\,\Delta x_i\cdot\Delta y_j\right)^2 + \left(f'_y(\xi_{ij},\eta_{ij})(\Delta x_i\cdot\Delta y_j)\right)^2 + \left(-\Delta x_i\cdot\Delta y_j\right)^2} =$$

$$\sqrt{1+\left(f'_x(\xi_{ij},\eta_{ij})\right)^2+\left(f'_y(\xi_{ij},\eta_{ij})\right)^2}\cdot\Delta x_i\cdot\Delta y_j$$

Tehát $\Delta F_{ij} \approx \sqrt{1+\left(f'_x(\xi_{ij},\eta_{ij})\right)^2+\left(f'_y(\xi_{ij},\eta_{ij})\right)^2}\,\Delta x_i\cdot\Delta y_j$

$$F \approx \sum_{(i,j)} \sqrt{1+\left(f'_x(\xi_{ij},\eta_{ij})\right)^2+\left(f'_y(\xi_{ij},\eta_{ij})\right)^2}\,\Delta x_i\cdot\Delta y_j$$

Tudjuk: $\iint_D f(x,y)\,dx\,dy \approx \sum_{(i,j)} f(\xi_{ij},\eta_{ij})\,\Delta x_i\cdot\Delta y_j$

$$F \approx \iint_D \sqrt{1+(f'_x)^2+(f'_y)^2}\,dx\,dy$$

$$\boxed{F = \iint_D \sqrt{1+(f'_x)^2+(f'_y)^2}\,dx\,dy}$$

Pl. 1. $z=x+y$ sík $D=\{(x,y):\ 0\le x\le 1,\ 0\le y\le 2\}$

feletti része.

$f(x,y)=x+y$, $f'_x=1$, $f'_y=1$

$$F=\iint_D \sqrt{1+1^2+1^2}\,dx\,dy = \int_{x=0}^{1}\Big(\underbrace{\int_{y=0}^{2}\sqrt{3}\,dy}_{\left[\sqrt{3}y\right]_0^{y=2}=\sqrt{3}\cdot 2}\Big)dx = \int_{x=0}^{1}\sqrt{3}\cdot 2\,dx = \left[\sqrt{3}\cdot 2x\right]_0^1 = \sqrt{3}\cdot 2$$

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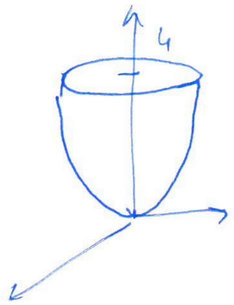
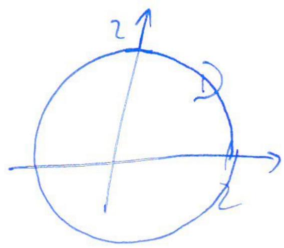
2. $z = x^2 + y^2 = f(x,y)$



$F = ?$ $\quad f'_x = 2x$ $\quad f'_y = 2y$

D: $x^2 + y^2 = 4$



$$F = \iint_D \sqrt{1 + 4x^2 + 4y^2}\, dx\, dy =$$

$x = r\cos\varphi$ $\quad 0 \le r \le 2$

$y = r\sin\varphi$ $\quad 0 \le \varphi \le 2\pi$

$$= \int_{r=0}^{2} \left(\int_{\varphi=0}^{2\pi} \sqrt{1+4r^2} \cdot r\, d\varphi \right) dr = \int_{r=0}^{2} \sqrt{1+4r^2} \cdot r\, 2\pi\, dr =$$

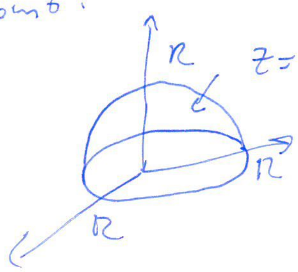
$$\left[\sqrt{1+4r^2} \cdot r \cdot \varphi \right]_0^{\varphi = 2\pi} = \sqrt{1+4r^2} \cdot r \cdot 2\pi$$

$$\int_{r=0}^{2} \sqrt{1+4r^2} \cdot 8r \cdot \frac{2\pi}{8}\, dr = \left[\frac{2\pi}{8} \frac{(1+4r^2)^{3/2}}{3/2} \right]_0^{r=2} = \frac{2\pi}{12} \left(17^{3/2} - 1^{3/2} \right) =$$

$$= \frac{2\pi \left(17^{3/2} - 1 \right)}{12}$$

3. $x^2 + y^2 + z^2 = R^2$ gömb felszíne

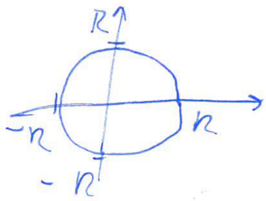
félgömb:



$z = \sqrt{R^2 - (x^2 + y^2)} = f(x,y)$

$$F = 2 \iint_D \sqrt{1 + (f'_x)^2 + (f'_y)^2}\, dx\, dy \overset{*}{=}$$

D:



$x = r\cos\varphi$ $\quad 0 \le r \le R$

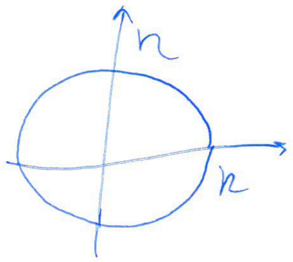
$y = r\sin\varphi$ $\quad 0 \le \varphi \le 2\pi$

$$f'_x = \frac{-2x}{2\sqrt{R^2 - x^2 - y^2}} = -\frac{x}{\sqrt{R^2 - x^2 - y^2}}$$

$$f'_y = \frac{-2y}{2\sqrt{R^2 - x^2 - y^2}} = -\frac{y}{\sqrt{R^2 - x^2 - y^2}}$$

$$\overset{*}{=} 2 \iint_D \sqrt{1 + \frac{x^2}{R^2 - x^2 - y^2} + \frac{y^2}{R^2 - x^2 - y^2}}\, dx\, dy = 2 \iint_D \sqrt{\frac{R^2 - x^2 - y^2 + x^2 + y^2}{R^2 - x^2 - y^2}}\, dx\, dy$$

$$= 2 \iint_D \sqrt{\frac{R^2}{R^2 - x^2 - y^2}}\, dx\, dy = 2 \iint_D \frac{R}{\sqrt{R^2 - (x^2 + y^2)}}\, dx\, dy$$



$x = r\cos\varphi \quad 0 \le r \le R$

$y = r\sin\varphi \quad 0 \le \varphi \le 2\pi$

$$= 2\int_{r=0}^{R}\left(\int_{\varphi=0}^{2\pi}\frac{R}{\sqrt{R^2-r^2}}\,r\,d\varphi\right)dr = 2\int_{r=0}^{R}\frac{2\pi R\cdot r}{\sqrt{R^2-r^2}}\,dr =$$

$$2\int_{r=0}^{R}2\pi R\,r\,(R^2-r^2)^{-1/2}\,dr = -2\pi R\int_{r=0}^{R}(R^2-r^2)^{-1/2}(-2r)\,dr = -2\pi R\left[\frac{(R^2-r^2)^{1/2}}{1/2}\right]_0^{r=R} =$$

$$-4\pi R\left((R^2-R^2)^{1/2}-(R^2-0^2)^{1/2}\right) = 4\pi R^2$$

Statikai nyomatékok, tömegközéppont

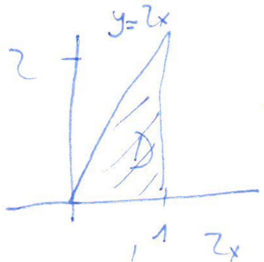
Feladat: Legyen a $D \subset \mathbb{R}^2$ vékony lemez sűrűségfv $f(x,y)$.
Határozzuk meg D tömegközéppontját!

Statikai nyomatékok: $m_x = \iint_D y\,f(x,y)\,dx\,dy$

$m_y = \iint_D x\,f(x,y)\,dx\,dy$

Tömegközéppont: $\left(\frac{m_y}{m}, \frac{m_x}{m}\right)$, ahol a tömeg: $m = \iint_D f(x,y)\,dx\,dy$.

Pl.



$f(x,y) = x+y$

$$m = \int_{x=0}^{1}\Big(\underbrace{\int_{y=0}^{2x} x+y\,dy}_{\left[xy+\frac{y^2}{2}\right]_0^{y=2x} = 2x^2+2x^2 = 4x^2}\Big)dx = \int_{x=0}^{1}4x^2\,dx = \left[\frac{4}{3}x^3\right]_0^1 = \frac{4}{3}$$

$$m_y = \int_{x=0}^{1} \left(\int_{y=0}^{2x} x(x+y)\,dy \right) dx = \int_{x=0}^{1} 4x^3\,dx = \left[x^4 \right]_0^1 = 1$$

$x^2 + xy$

$$\left[x^2 y + x\frac{y^2}{2} \right]_0^{y=2x} = 2x^3 + 2x^3 = 4x^3$$

$$m_x = \int_{x=0}^{1} \left(\int_{y=0}^{2x} y(x+y)\,dy \right) dx = \int_{x=0}^{1} \frac{14}{3} x^3\,dx = \left[\frac{14}{12} x^4 \right]_0^1 = \frac{7}{6}$$

$xy + y^2$

$$\left[x\frac{y^2}{2} + \frac{y^3}{3} \right]_0^{y=2x} = 2x^3 + \frac{8}{3}x^3 = \frac{14}{3}x^3$$

Tömegközéppont: $\left(\frac{1}{4/3} ; \frac{7/6}{4/3} \right) = \left(\frac{3}{4}, \frac{21}{24} \right) = \left(\frac{3}{4}, \frac{7}{8} \right)$