1.

a) $f(x) = \cos(3x)\, e^{-x} \rightsquigarrow f(0) = 1$

$$f'(x) = -3\sin(3x)\, e^{-x} - \cos(3x)\, e^{-x} \rightsquigarrow f'(0) = -1$$

$$f''(x) = -9\cos(3x)\, e^{-x} + 3\sin(3x)\, e^{-x}$$

$$+ 3\sin(3x)\, e^{-x} + \cos(3x)\, e^{-x} =$$

$$= -8\cos(3x)\, e^{-x} + 6\sin(3x)\, e^{-x} \rightsquigarrow f''(0) = -8$$

$$f'''(x) = 24\sin(3x)\, e^{-x} + 8\cos(3x)\, e^{-x}$$

$$+ 18\cos(3x)\, e^{-x} - 6\sin(3x)\, e^{-x} =$$

$$= 18\sin(3x)\, e^{-x} + 26\cos(3x)\, e^{-x} \rightsquigarrow f'''(0) = 26$$

A harmadrendű Taylor-polinom a 0-ban:

$$T^3_{f,0}(x) = \frac{1}{0!}x^0 + \frac{-1}{1!}x^1 + \frac{-8}{2!}x^2 + \frac{26}{3!}x^3 =$$

$$= 1 - x + 2x^2 + \frac{13}{3}x^3$$

b) $g(x) = \sin\left(\frac{\pi}{3} - 2x\right) \rightsquigarrow g(0) = \frac{\sqrt{3}}{2}$

$g'(x) = -2\cos\left(\frac{\pi}{3} - 2x\right) \rightsquigarrow g'(0) = -1$

$g''(x) = -4\sin\left(\frac{\pi}{3} - 2x\right) \rightsquigarrow g''(0) = -2\sqrt{3}$

$g'''(x) = 8\cos\left(\frac{\pi}{3} - 2x\right) \rightsquigarrow g'''(0) = 4$

$$T^3_{g,0}(x) = \frac{\frac{\sqrt{3}}{2}}{0!}x^0 + \frac{-1}{1!}x^1 + \frac{-2\sqrt{3}}{2!}x^2 + \frac{4}{3!}x^3 =$$

$$= \boxed{\frac{\sqrt{3}}{2} - x - \sqrt{3}\,x^2 + \frac{2}{3}x^3}$$

c

$$h(x) = \operatorname{arctg}(x^2) \rightsquigarrow h(1) = \frac{\sqrt{2}}{2}$$

$$h'(x) = \frac{2x}{x^4+1} \rightsquigarrow h'(x) = 1$$

$$h''(x) = \frac{2(x^4+1) - 2x \cdot 4x^3}{(x^4+1)^2} =$$

$$= \frac{2-6x^4}{(x^4+1)^2} \rightsquigarrow h''(1) = -1$$

$$h'''(x) = \frac{-24x^3(x^4+1)^2 - (2-6x^4)2(x^4+1)4x^3}{(x^4+1)^4} =$$

$$= \frac{8x^3(3x^4-5)}{(x^4+1)^3} \rightsquigarrow h'''(1) = -2$$

$$T^3_{h,1}(x) = \frac{\sqrt{2}}{2} + (x-1) - \frac{1}{2}(x-1)^2 - \frac{1}{3}(x-1)^3$$

2.

a) $$\lim_{x\to 1} \frac{\ln(x)}{x-1} \overset{L'H}{=} \lim_{x\to 1} \frac{\frac{1}{x}}{1} = \lim_{x\to 1} \frac{1}{x} = \boxed{1}$$

b) $$\lim_{x\to\infty} \frac{x\cdot e^{\frac{x}{2}}}{e^x + x} \overset{L'H}{=} \lim_{x\to\infty} \frac{e^{\frac{x}{2}} + \frac{1}{2}\cdot x\cdot e^{\frac{x}{2}}}{e^x + 1} \overset{L'H}{=}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{2}\cdot e^{\frac{x}{2}} + \frac{1}{2} e^{\frac{x}{2}} + \frac{1}{4} x\cdot e^{\frac{x}{2}}}{e^x} =$$

$$= \lim_{x\to\infty} \frac{1}{2}\cdot \underbrace{e^{-\frac{x}{2}}}_{\to 0} + \frac{1}{2}\underbrace{e^{-\frac{x}{2}}}_{\to 0} + \frac{1}{4}\, x\cdot e^{-\frac{x}{2}} =$$

$$= \frac{1}{4} \lim_{x\to\infty} x\cdot e^{-\frac{x}{2}} = \frac{1}{4}\cdot \lim_{x\to\infty} \frac{x}{e^{\frac{x}{2}}} \overset{L'H}{=}$$

$$= \frac{1}{4} \lim_{x\to\infty} \frac{1}{\frac{1}{2} e^{\frac{x}{2}}} = \frac{1}{2}\cdot \lim_{x\to\infty} e^{-\frac{x}{2}} = \boxed{0}.$$

c)

$$\lim_{x\to\infty} x^2 \cdot e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x} \overset{L'H}{=} \lim_{x\to\infty} \frac{2x}{e^x} \overset{L'H}{=}$$

$$\lim_{x\to\infty} \frac{2}{e^x} = \lim_{x\to\infty} 2e^{-x} = \boxed{0}.$$

d)

$$\lim_{x\to 0+} (\sin(x))^x = \lim_{x\to 0+} e^{\ln(\sin(x))\cdot x} = e^0 = \boxed{1},$$

mivel teljesül, hogy

$$\lim_{x\to 0+} \ln(\sin(x))\cdot x = \lim_{x\to 0+} \frac{\ln(\sin(x))}{\frac{1}{x}} =$$

$$\overset{L'H}{=} \lim_{x\to 0+} \frac{\frac{\cos(x)}{\sin(x)}}{-\frac{1}{x^2}} = \lim_{x\to 0+} \underbrace{\frac{x}{\sin(x)}}_{\to 1} \cdot \frac{\overbrace{-x}^{\to 0}}{\underbrace{\cos(x)}_{\to 1}} = 0,$$

továbbá $x \mapsto e^x$ folytonos függvény.

e

$$\lim_{x\to 0} \frac{\overbrace{e^x - e^{-x} - 2x}^{\to 0}}{\underbrace{x - \sin(x)}_{\to 0}} \overset{L'H}{=} \lim_{x\to 0} \frac{\overbrace{e^x + e^x - 2}^{\to 0}}{\underbrace{1 - \cos(x)}_{\to 0}} \overset{L'H}{=}$$

$$= \lim_{x\to 0} \frac{e^x + e^x}{\sin(x)}$$ nem létezik, mivel

$$\lim_{x\to 0+} \frac{\overbrace{e^x + e^x}^{\to 2}}{\underbrace{\sin(x)}_{\to 0+}} = +\infty$$, azonban

$$\lim_{x\to 0-} \frac{\overbrace{e^x + e^x}^{\to 2}}{\underbrace{\sin(x)}_{\to 0-}} = -\infty .$$

Vagyis

$$\lim_{x\to 0+} \frac{e^x - e^{-x} - 2x}{x - \sin(x)} = +\infty$$ és $$\lim_{x\to 0-} \frac{e^x - e^{-x} - 2x}{x - \sin(x)} = -\infty ,$$

így

$$\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin(x)}$$ nem létezik.

(f)

$$\lim_{x\to 0_+} x^x = \lim_{x\to 0_+} e^{\ln(x)\cdot x} = e^0 = \boxed{1},$$

hiszen

$$\lim_{x\to 0+} \ln(x)\cdot x = \lim_{x\to 0+} \frac{\overset{\to -\infty}{\ln(x)}}{\underset{\to +\infty}{\frac{1}{x}}} \overset{L'H}{=} \lim_{x\to 0+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = 0.$$