1.

a) • $z_1 + z_2 = (2+5i) + (-3i+4)$

$= (2+4) + (5+(-3))i = 6+2i$

$z_1 - z_2 = (2+5i) - (-3i+4)$

$= (2-4) + (5-(-3))i = -2+8i$

• $z_1 \cdot z_2 = (2+5i) \cdot (-3i+4)$

$= (2\cdot 4 - 5\cdot(-3)) + (2\cdot(-3) + 5\cdot 4)i$

$= (8+15) + (-6+20)i = 23+14i$

• $\frac{z_1}{z_2} = \frac{2+5i}{-3i+4} = \frac{2+5i}{-3i+4} \cdot \frac{3i+4}{3i+4}$

$= \frac{(2\cdot 4 - 5\cdot 3) + (5\cdot 4 + 3\cdot 2)i}{3^2+4^2}$

$= \frac{(8-15) + (20+6)i}{9+16}$

$$= \frac{-7+26i}{25} = -\frac{7}{25} + \frac{26}{25}i$$

- $z_1(z_2 - z_1) = -z_1(z_1 - z_2)$

$$= (-2-5i)\cdot(-2+8i) = 44 - 6i$$

- $\frac{z_2^2}{\overline{z_1}} = \frac{z_2^2 \cdot z_1}{\overline{z_1}\cdot z_1} = \frac{(4-3i)^2(2+5i)}{2^2+5^2}$

$$= \frac{(16-24i+3i^2)(2+5i)}{29}$$

$$= \frac{(13-24i)(2+5i)}{29}$$

$$= \frac{26+65i-48i-120i^2}{29}$$

$$= \frac{146+17i}{29} = \frac{146}{29} + \frac{17}{29}i$$

b) • $z_1 + z_2 = (\sqrt{3} - i) + (1 + i) =$

$= (\sqrt{3} + 1) + (-1 + 1) i = \sqrt{3} + 1.$

$z_1 - z_2 = (\sqrt{3} - i) - (1 + i) =$

$= (\sqrt{3} - 1) + (-1 - 1) i = \sqrt{3} - 1 - 2i$

• $z_1 \cdot z_2 = (\sqrt{3} - i) \cdot (1 + i) =$

$= \sqrt{3} + \sqrt{3} i - i - i^2 = \sqrt{3} + 1 + (\sqrt{3} - 1) i$

• $\frac{z_1}{z_2} = \frac{\sqrt{3} - i}{1 + i} = \frac{\sqrt{3} - i}{1 + i} \cdot \frac{1 - i}{1 - i} = \frac{(\sqrt{3} - i)(1 - i)}{1^2 + 1^2} =$

$= \frac{\sqrt{3} - \sqrt{3} i - i + i^2}{2} = \frac{\sqrt{3} - 1}{2} - \frac{\sqrt{3} + 1}{2} i$

• $z_1 (z_2 - z_1) = -z_1 (z_1 - z_2) =$

$= (-\sqrt{3} + i)(\sqrt{3} - 1 - 2i) = -3 + 2$

$= -\sqrt{3}(\sqrt{3} - 1) + 2\sqrt{3} i + (\sqrt{3} - 1) i - 2i^2$

$$= \sqrt{3} - 1 + (3\sqrt{3} - 1)i$$

- $$\frac{z_2^2}{z_1} = \frac{z_2^2 \cdot \bar{z}_1}{z_1 \cdot \bar{z}_1} = \frac{z_2 (z_2 \cdot \bar{z}_1)}{|z_1|^2} =$$

$$= \frac{(1+i)\,(\sqrt{3}+1 + (\sqrt{3}-1)i)}{\sqrt{3}^2 + 1^2}$$

$$= \frac{\sqrt{3}+1 + (\sqrt{3}-1)i + (\sqrt{3}+1)i + (\sqrt{3}-1)i^2}{3+1}$$

$$= \frac{\sqrt{3}+1 + (\sqrt{3}-1)i + (\sqrt{3}+1)i + (\sqrt{3}-1)i^2}{3+1}$$

$$= \frac{\sqrt{3}+1 + 2\sqrt{3}i - (\sqrt{3}-1)}{4}$$

$$= \frac{2 + 2\sqrt{3}i}{4} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

c) • $z_1 + z_2 = (1-3i) + (-i) =$

$= (1+0) + (-3-1)i = 1-4i$.

• $z_1 - z_2 = (1-3i) - (-i) =$

$= (1-0) + (-3+1)i = 1-2i$

• $z_1 \cdot z_2 = (1-3i) \cdot (-i) =$

$= -i + 3i^2 = -3-i$

• $\dfrac{z_1}{z_2} = \dfrac{1-3i}{-i} = \dfrac{1-3i}{-i} \cdot \dfrac{i}{i} = \dfrac{(1-3i)i}{1^2} =$

$= \dfrac{i-3i^2}{1} = 3+i$

• $z_1(z_2 - z_1) = -z_1(z_1 - z_2) =$

$= (-1+3i)(1-2i) = -1+2i+3i-6i^2$

$= 5+5i$

$$\frac{z_2^2}{\overline{z}_1} = \frac{z_2^2 \cdot z_1}{\overline{z}_1 \cdot z_1} = \frac{z_2 \overbrace{(z_2 \cdot z_1)}^{-3-i}}{|z_1|^2} =$$

$$= \frac{i^2(-3-i)}{1^2+3^2} = \frac{(-1)(-3-i)}{10} = \frac{3}{10} + \frac{1}{10} i$$

d) • $z_1 + z_2 = (2-i) + (-4+7i) = -2+6i$

• $z_1 - z_2 = (2-i) - (-4+7i) = 6-8i$

• $z_1 \cdot z_2 = (2-i)(-4+7i) = -1+18i$

• $$\frac{z_1}{z_2} = \frac{2-i}{-4+7i} = \frac{2-i}{-4+7i} \cdot \frac{-4-7i}{-4-7i} = \frac{-15-10i}{4^2+7^2} =$$

$$= -\frac{3}{13} - \frac{2}{13} i$$

• $z_1(z_2 - z_1) = 4+22i$

• $$\frac{z_2^2}{\overline{z}_1} = \frac{z_2^2 \cdot z_1}{\overline{z}_1 \cdot z_1} = \frac{z_2(z_2 \cdot z_1)}{2^2+1^2} = \frac{(-4+7i)(-1+18i)}{5}$$

$$= \frac{4-72i-7i+136i^2}{5} = -\frac{132}{5} - \frac{79}{5} i$$

2.

a) $(1+6i)-i(-4+5i)=(1+6i)$

$+4i-5i^2 = 1+6i+4i+5 =$

$= 6+10i$

b) $(1+i)\overline{(2-3i)} = (1+i)(2+3i) =$

$=(2-3)+(3+2)i = -1+5i$

c) $\overline{(2+i)}(4-7i) = (2-i)(4-7i) =$

$=(8-7)+(-14-4)i = 1+18i$

d) $\dfrac{2+4i}{3-2i} = \dfrac{(2+4i)(3+2i)}{(3-2i)(3+2i)} =$

$= \dfrac{(6-8)+(4+12)i}{3^2+2^2} = -\dfrac{2}{13}+\dfrac{16}{13}i$

e) $\frac{1}{(1-i)^2} = \frac{1}{1-2i+i^2} = \frac{1}{-2i} = \frac{1}{-2i} \cdot \frac{2i}{2i} =$

$= \frac{2i}{-4i^2} = \frac{2i}{4} = \frac{1}{2} i$

f) $\frac{2+i}{i(1-4i)} = \frac{2+i}{4+i} = \frac{2+i}{4+i} \cdot \frac{4-i}{4-i} =$

$= \frac{(8+1)+(4-2)i}{4^2+1^2} = \frac{9}{17} + \frac{2}{17} i$

3. Vegyük észre, hogy

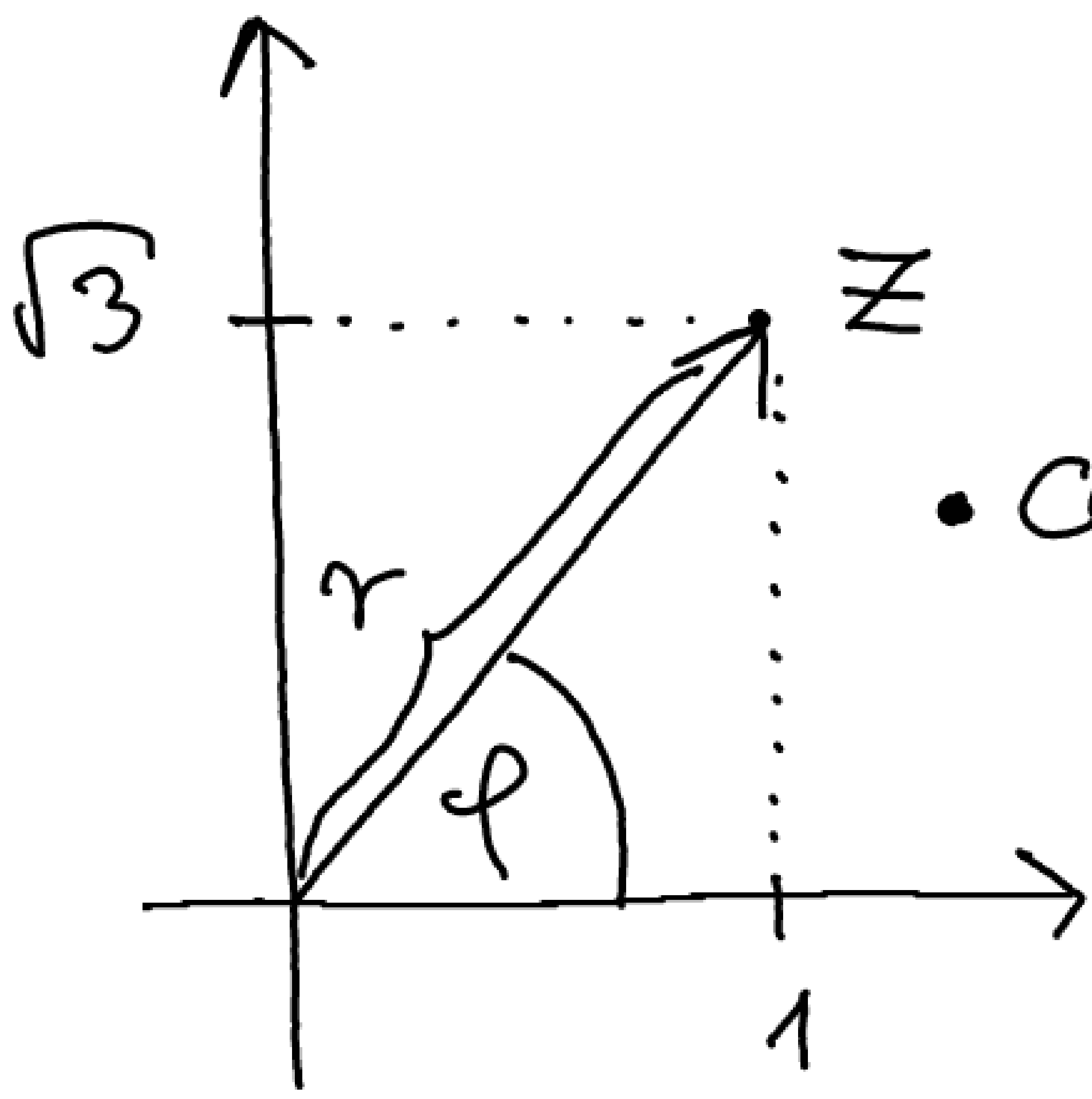
$i^1 = i \Rightarrow i^2 = -1 \Rightarrow i^3 = -i \Rightarrow i^4 = 1 \Rightarrow i^5 = i.$

Vagyis 4 periódusonként ismétlődni fog. Azt kapjuk, hogy

$$i^n = \begin{cases} 1, & \text{ha } n = 4k \\ i, & \text{ha } n = 4k+1 \\ -1, & \text{ha } n = 4k+2 \\ -i, & \text{ha } n = 4k+3. \end{cases}$$

4.

a) $z = 1 + i\sqrt{3}$



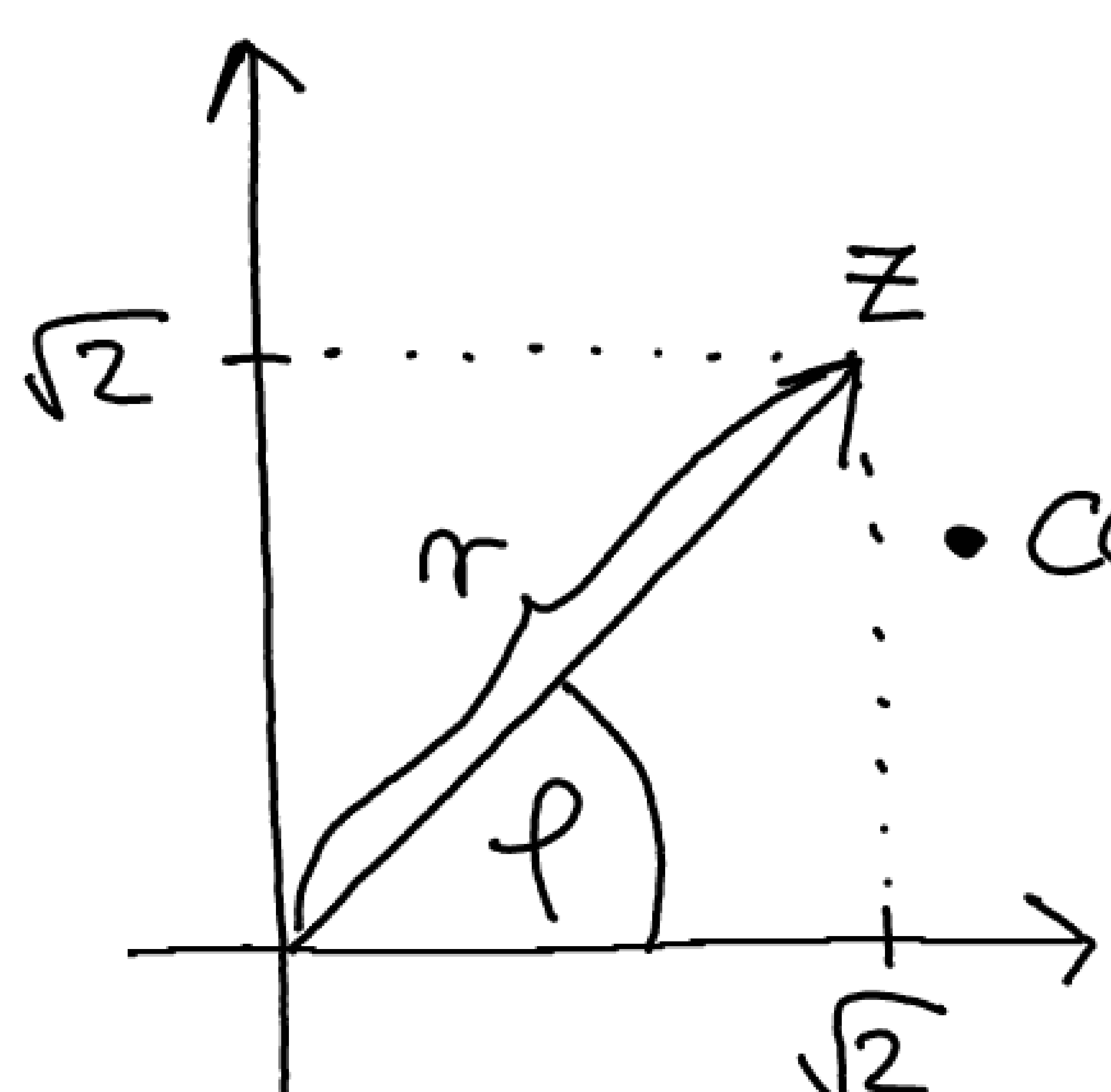
- $r = \sqrt{1^2 + \sqrt{3}^2} = \sqrt{4} = 2$
- $\cos(\varphi) = \frac{1}{2}$, amiből pedig

$$\varphi = \frac{\pi}{3}$$

A trigonometrikus alak innen

$$z = 2 \cdot \left(\cos\left(\frac{\pi}{3}\right) + i \cdot \sin\left(\frac{\pi}{3}\right) \right)$$

b) $z = \sqrt{2} + i\sqrt{2}$



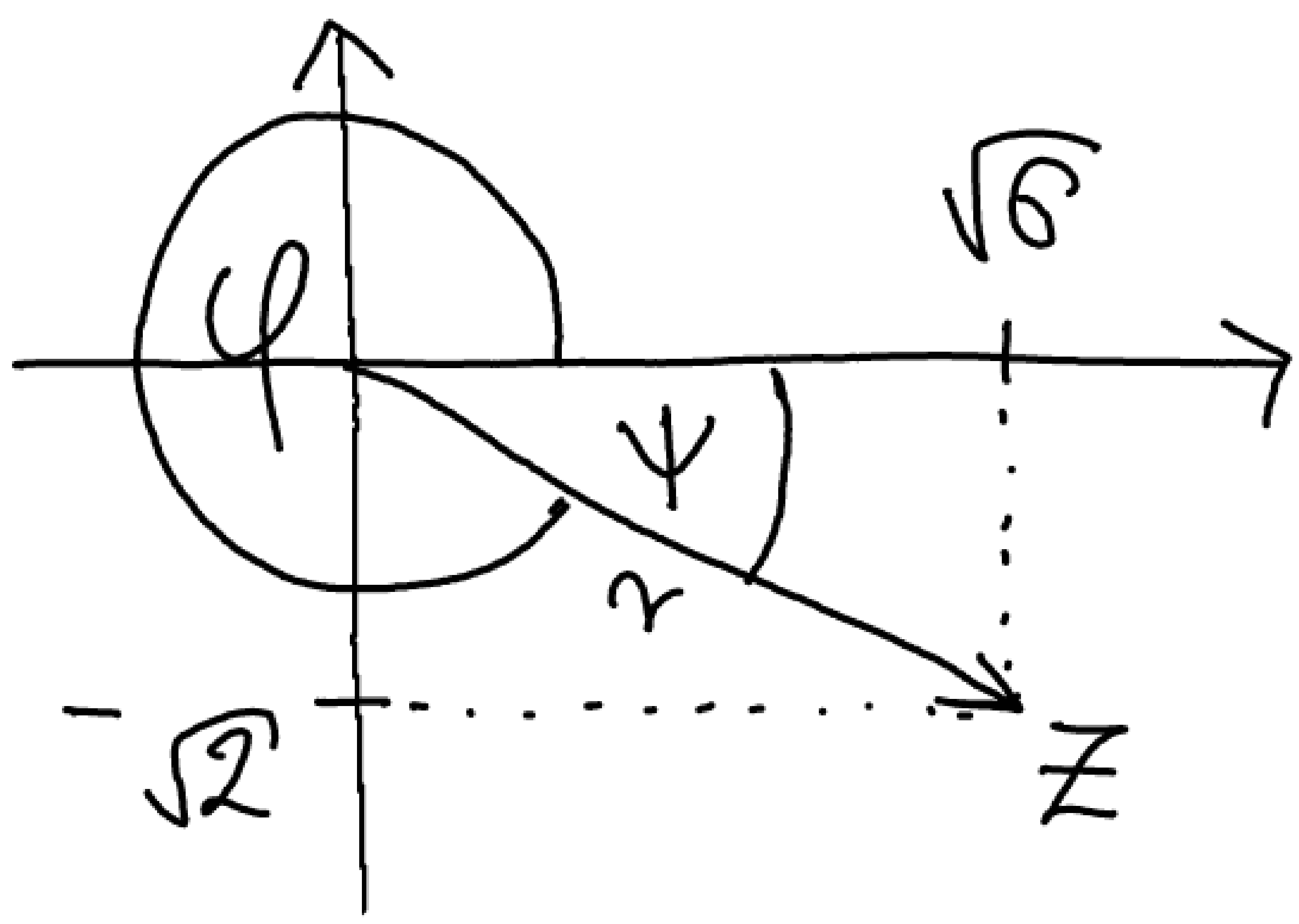
- $r = \sqrt{\sqrt{2}^2 + \sqrt{2}^2} = \sqrt{4} = 2$
- $\cos(\varphi) = \frac{\sqrt{2}}{2}$, amiből pedig

$$\varphi = \frac{\pi}{4}$$

A trigonometrikus alak innen

$$z = 2 \cdot \left(\cos\left(\frac{\pi}{4}\right) + i \cdot \sin\left(\frac{\pi}{4}\right) \right)$$

c) $z = \sqrt{6} - i\sqrt{2}$



$r = \sqrt{\sqrt{6}^2 + \sqrt{2}^2} = 2\sqrt{2}$

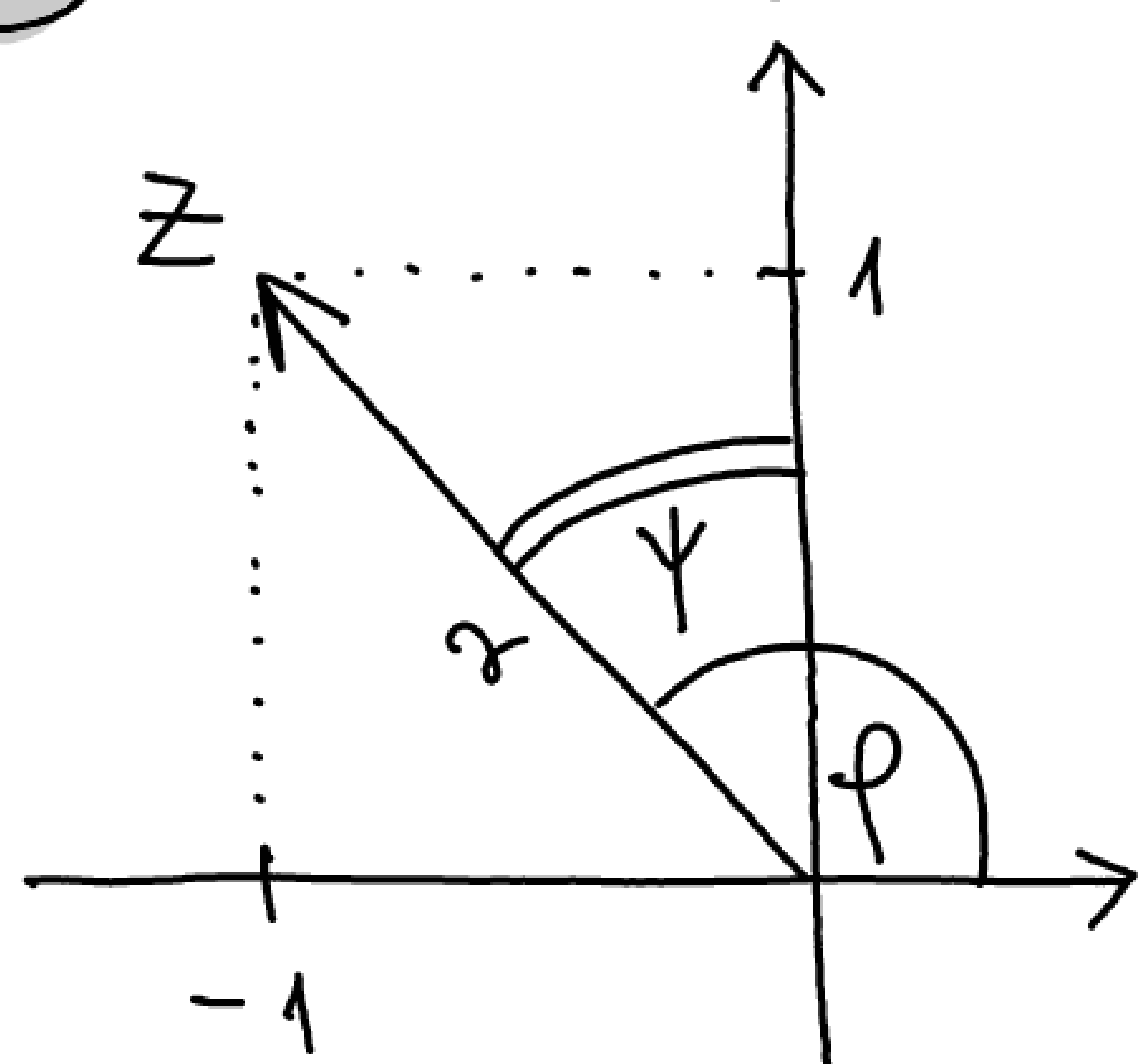
$\sin(\psi) = \dfrac{\sqrt{2}}{2\sqrt{2}} = \dfrac{1}{2}$,

amiből $\psi = \dfrac{\pi}{6}$.

Innen pedig $\varphi = 2\pi - \psi = 2\pi - \dfrac{\pi}{6} = \dfrac{11\pi}{6}$.

$z = 2\sqrt{2}\left(\cos\left(\dfrac{11\pi}{6}\right) + i \cdot \sin\left(\dfrac{11\pi}{6}\right)\right)$.

d) $z = -1 + i$



$r = \sqrt{1^2 + 1^2} = \sqrt{2}$

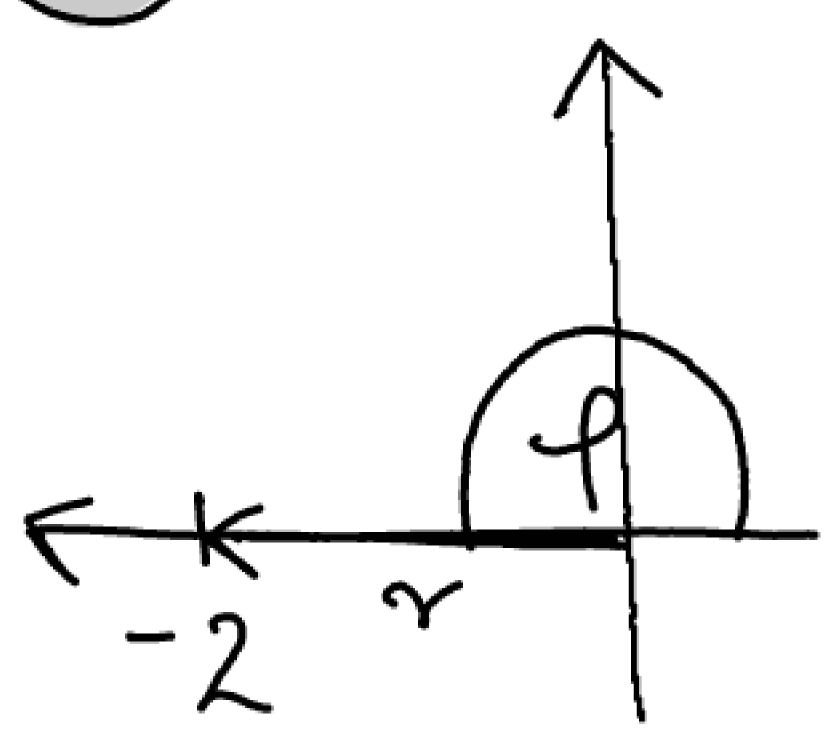
$\cos(\psi) = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$,

amiből $\psi = \dfrac{\pi}{4}$.

Innen pedig $\varphi = \dfrac{\pi}{2} + \psi = \dfrac{\pi}{2} + \dfrac{\pi}{4} = \dfrac{3\pi}{4}$

$z = \sqrt{2}\left(\cos\left(\dfrac{3\pi}{4}\right) + i \cdot \sin\left(\dfrac{3\pi}{4}\right)\right)$.

(e) $z=-2$

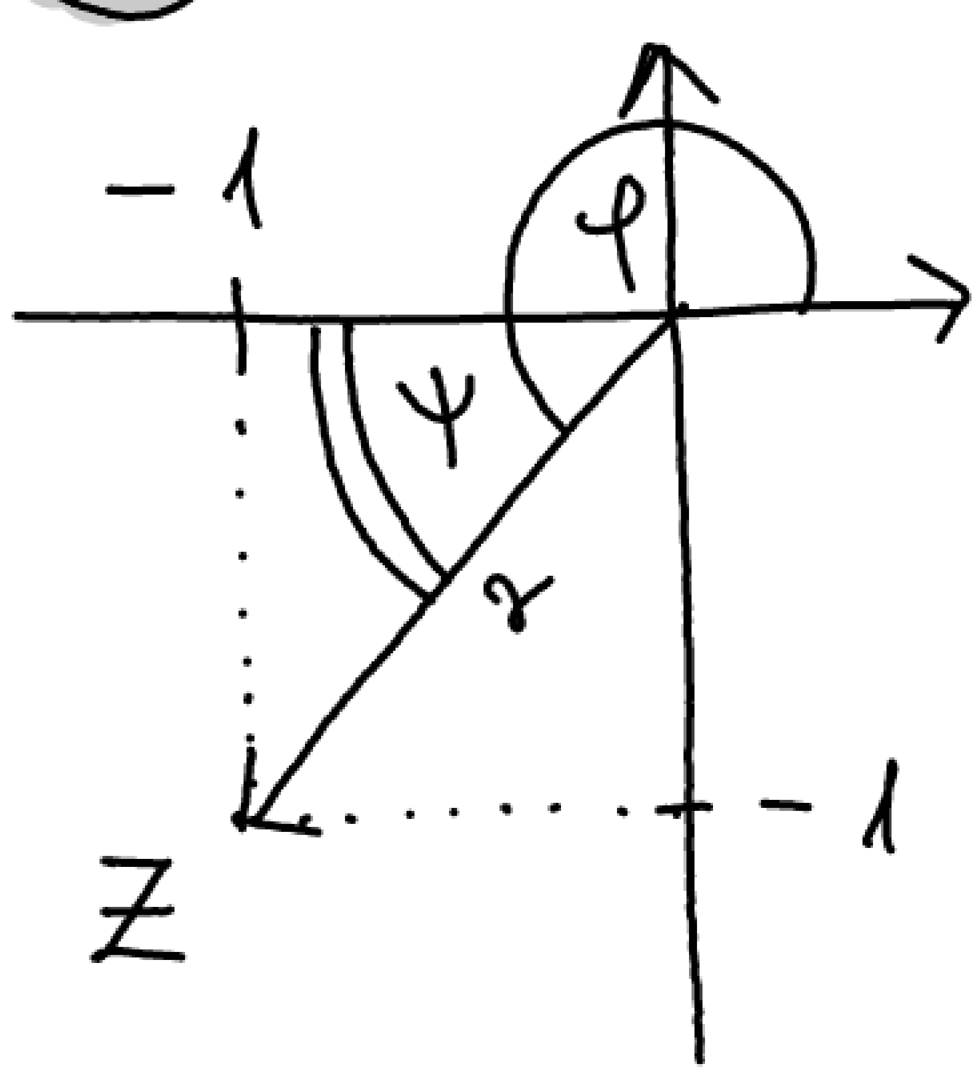


Egyből látszik, hogy

$r=2$ és $\varphi=\pi$.

$$z=2\left(\cos(\pi)+i\cdot\sin(\pi)\right)$$

(f) $z=-1-i$



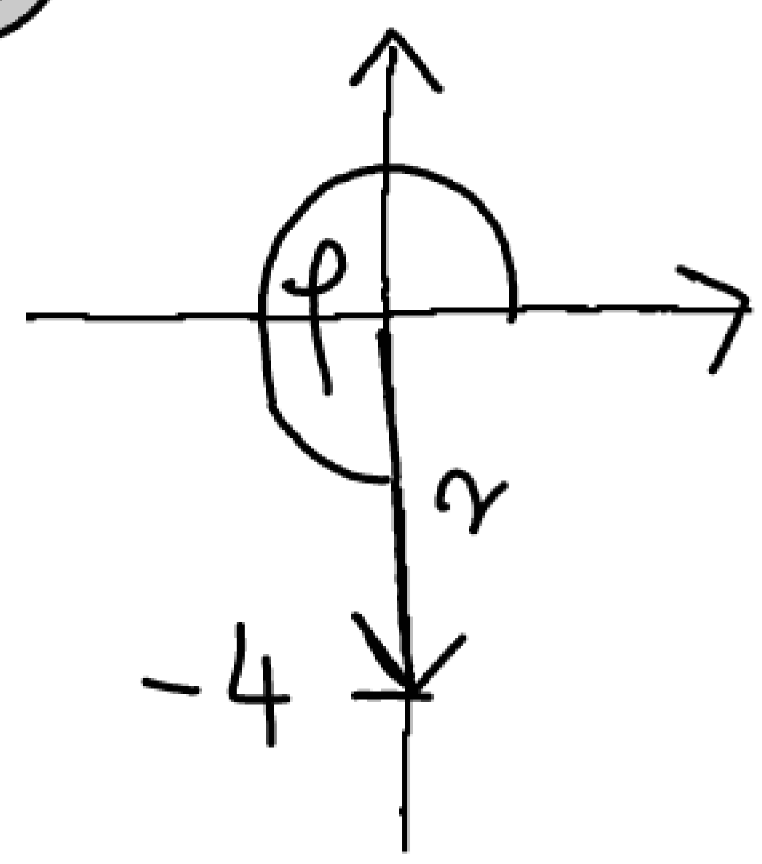
Könnyen látható, hogy

$r=\sqrt{2}$ és $\psi=\frac{\pi}{4}$,

amiből $\varphi=\pi+\psi=\pi+\frac{\pi}{4}=\frac{5\pi}{4}$

$$z=\sqrt{2}\left(\cos\left(\frac{5\pi}{4}\right)+i\cdot\sin\left(\frac{5\pi}{4}\right)\right)$$

g) $z = -4i$

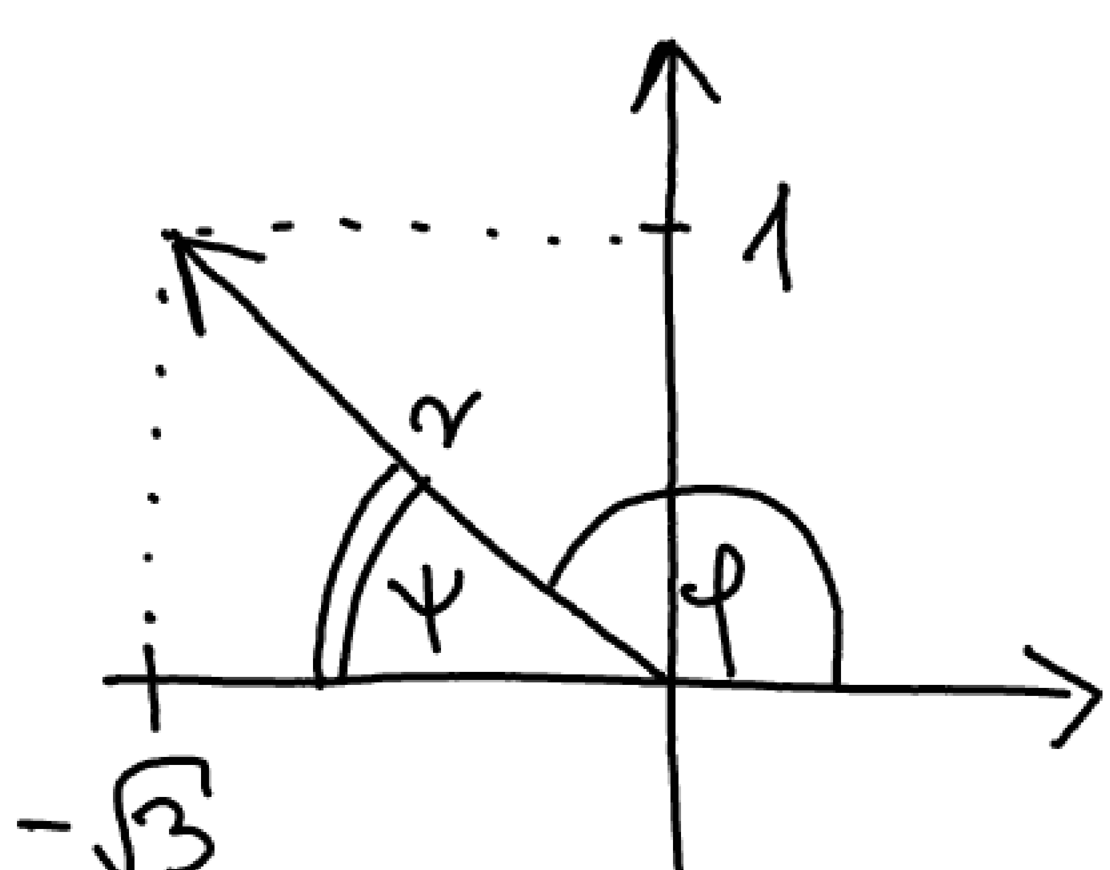


Ebben az esetben is egyből látszik, hogy

$$r = 4 \text{ és } \varphi = \frac{3\pi}{2}$$

$$z = 4\left(\cos\left(\frac{3\pi}{2}\right) + i\cdot\sin\left(\frac{3\pi}{2}\right)\right)$$

h) $z = -\sqrt{3} + i$



$$r = \sqrt{1^2 + \sqrt{3}^2} = \sqrt{4} = 2$$

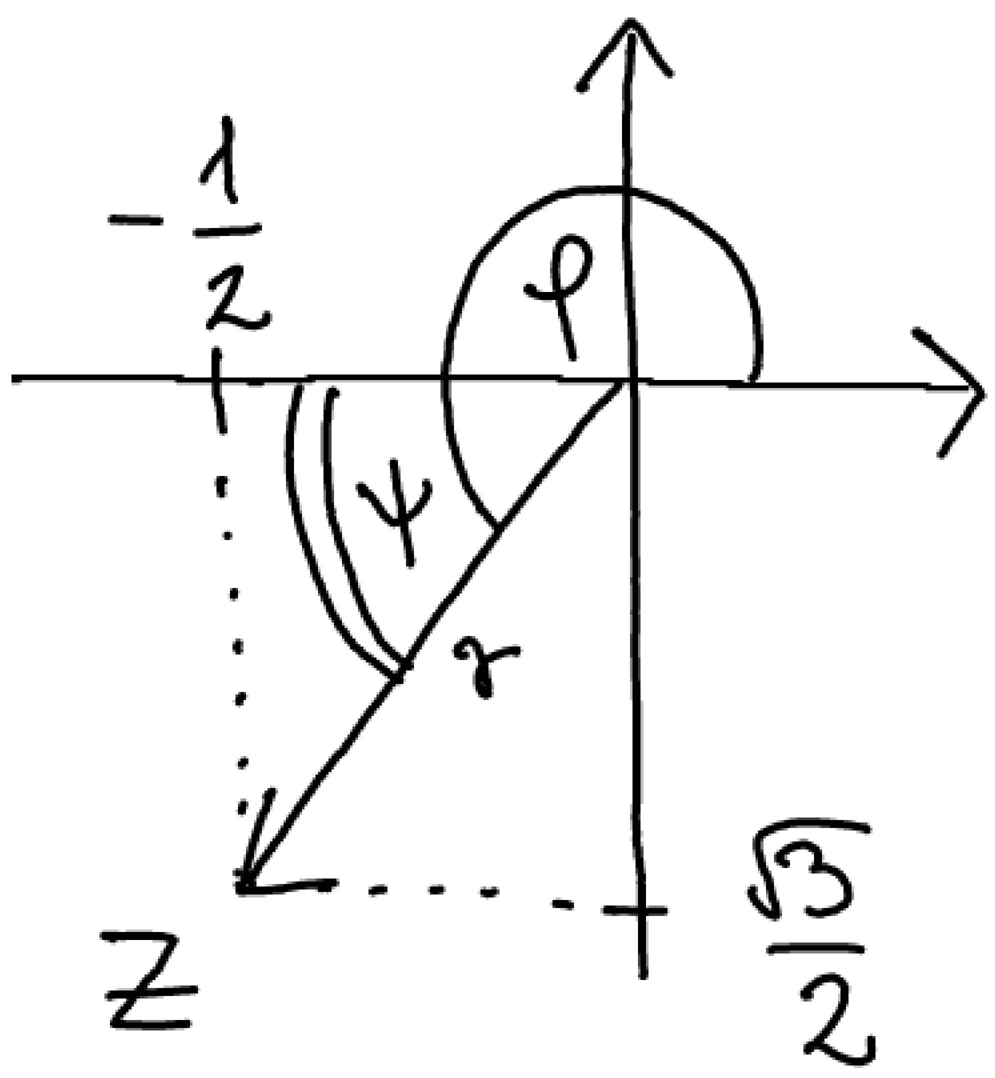
$\sin(\psi) = \frac{1}{2}$, ezért $\psi = \frac{\pi}{6}$,

amiből $\varphi = \pi - \psi = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

$$z = 2\left(\cos\left(\frac{5\pi}{6}\right) + i\cdot\sin\left(\frac{5\pi}{6}\right)\right)$$

(i) $z = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$

$$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{1} = 1$$



$\sin(\psi) = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2}$, amiből

$\psi = \frac{\pi}{3}$,

innen pedig $\varphi = \pi + \psi = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$.

$$z = \cos\left(\frac{4\pi}{3}\right) + i \cdot \sin\left(\frac{4\pi}{3}\right).$$

5.

(a) $4\left(\cos\left(\frac{\pi}{6}\right) + i \cdot \sin\left(\frac{\pi}{6}\right)\right) =$

$= 4 \cdot \cos\left(\frac{\pi}{6}\right) + 4 \sin\left(\frac{\pi}{6}\right) i =$

$= 4 \cdot \frac{\sqrt{3}}{2} + 4 \cdot \frac{1}{2} \cdot i = 2\sqrt{3} + 2i.$

b) $3\left(\cos\left(\frac{2\pi}{3}\right) + i \cdot \sin\left(\frac{2\pi}{3}\right)\right) =$

$= 3\cos\left(\frac{2\pi}{3}\right) + 3\sin\left(\frac{2\pi}{3}\right) i =$

$= 3 \cdot \left(-\frac{1}{2}\right) + 3 \cdot \frac{\sqrt{3}}{2} \; i = -\frac{3}{2} + \frac{3\sqrt{3}}{2} i$

c) $2\left(\cos\left(\frac{5\pi}{3}\right) + i \cdot \sin\left(\frac{5\pi}{3}\right)\right) =$

$= 2\cos\left(\frac{5\pi}{3}\right) + 2\sin\left(\frac{5\pi}{3}\right) i =$

$= 2 \cdot \frac{1}{2} + 2 \cdot \left(-\frac{\sqrt{3}}{2}\right) i = 1 - \sqrt{3} i$

6.

a) • $z_1 \cdot z_2 = 2 \cdot 4\left(\cos\left(\frac{\pi}{3}+\frac{\pi}{4}\right)+i\cdot\sin\left(\frac{\pi}{3}+\frac{\pi}{4}\right)\right)$

$$= 8\left(\cos\left(\frac{7\pi}{12}\right)+i\cdot\sin\left(\frac{7\pi}{12}\right)\right)$$

• $\frac{z_1}{z_2} = \frac{2}{4}\cdot\left(\cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)+i\cdot\sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)\right)$

$$= \frac{1}{2}\left(\cos\left(\frac{\pi}{12}\right)+i\cdot\sin\left(\frac{\pi}{12}\right)\right)$$

• Mivel $\overline{z_1} = 4\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right)$, ezért

$$\frac{z_2^2}{\overline{z_1}} = \frac{2^2}{4}\left(\cos\left(2\cdot\frac{\pi}{3}-\left(-\frac{\pi}{4}\right)\right)+i\cdot\sin\left(2\cdot\frac{\pi}{3}-\left(-\frac{\pi}{4}\right)\right)\right) =$$

$$= \cos\left(\frac{11\pi}{12}\right)+i\sin\left(\frac{11\pi}{12}\right).$$

b

- $z_1 \cdot z_2 = 2 \cdot 4 \left(\cos\left(\frac{\pi}{4} + \frac{2\pi}{3}\right) + i \cdot \sin\left(\frac{\pi}{4} + \frac{2\pi}{3}\right)\right)$

$= 8 \left(\cos\left(\frac{11\pi}{12}\right) + i \cdot \sin\left(\frac{11\pi}{12}\right)\right)$

- $\frac{z_1}{z_2} = \frac{2}{4} \cdot \left(\cos\left(\frac{\pi}{4} - \frac{2\pi}{3}\right) + i \cdot \sin\left(\frac{\pi}{4} - \frac{2\pi}{3}\right)\right) =$

$= \frac{1}{2} \left(\cos\left(-\frac{5\pi}{12}\right) + i \cdot \sin\left(-\frac{5\pi}{12}\right)\right) =$

$= \frac{1}{2} \left(\cos\left(2\pi - \frac{5\pi}{12}\right) + i \cdot \sin\left(2\pi - \frac{5\pi}{12}\right)\right) =$

$= \frac{1}{2} \left(\cos\left(\frac{19\pi}{12}\right) + i \cdot \sin\left(\frac{19\pi}{12}\right)\right)$

• Mivel $\overline{z}_1 = 4\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$, ezért

$$\frac{z_2^2}{\overline{z}_1} = \frac{2^2}{4}\left(\cos\left(2\cdot\frac{\pi}{4} - \left(-\frac{2\pi}{3}\right)\right) + i\cdot\sin\left(2\cdot\frac{\pi}{4} - \left(-\frac{2\pi}{3}\right)\right)\right)$$

$$= \cos\left(\frac{14\pi}{12}\right) + i\cdot\sin\left(\frac{14\pi}{12}\right)$$

7. A feladatban nem írtam le az algebrai alakból trigonometrikus alakba történő átváltás részleteit.

a) $$(1+\sqrt{3}i)^{10} = \left[2\left(\cos\left(\frac{\pi}{3}\right) + i\cdot\sin\left(\frac{\pi}{3}\right)\right)\right]^{10} =$$

$$= 2^{10}\left(\cos\left(\frac{10\pi}{3}\right) + i\cdot\sin\left(\frac{10\pi}{3}\right)\right) =$$

$$= 1024\left(\cos\left(\frac{4\pi}{3}\right) + i\cdot\sin\left(\frac{4\pi}{3}\right)\right) =$$

$$= 1024\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = -512 - 512\sqrt{3}\,i.$$

b) $$(1-i)^4 = \left[\sqrt{2}\left(\cos\left(\frac{5\pi}{4}\right) + i\cdot\sin\left(\frac{5\pi}{4}\right)\right)\right]^4 =$$

$$= \sqrt{2}^4\left(\cos\left(5\pi\right) + i\cdot\sin\left(5\pi\right)\right) =$$

$$= 4\left(\cos(\pi) + i\cdot\sin(\pi)\right) = 4\cdot(-1+0) = -4$$

c) $$(-1+i)^7 = \left[\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\cdot\sin\left(\frac{3\pi}{4}\right)\right)\right]^7 =$$

$$= \sqrt{2}^7\left(\cos\left(\frac{21\pi}{4}\right) + i\cdot\sin\left(\frac{21\pi}{4}\right)\right) =$$

$$= 8\sqrt{2}\left(\cos\left(\frac{5\pi}{4}\right) + i\cdot\sin\left(\frac{5\pi}{4}\right)\right) =$$

$$= 8\sqrt{2}\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) = -8 - 8i$$

d $(1+i)^{12} = \left[\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\cdot\sin\left(\frac{\pi}{4}\right)\right)\right]^{12} =$

$$= \sqrt{2}^{12}\left(\cos(3\pi) + i\cdot\sin(3\pi)\right) =$$

$$= 64(-1+0) = -64$$

e $(-2\sqrt{3}+2i)^{-9} = \left(\frac{1}{-2\sqrt{3}+2i}\right)^{9} =$

$$= \left[\frac{\cos(0) + i\cdot\sin(0)}{4\left(\cos\left(\frac{5\pi}{6}\right) + i\cdot\sin\left(\frac{5\pi}{6}\right)\right)}\right]^{9} =$$

$$= \left[\frac{1}{4}\left(\cos\left(-\frac{5\pi}{6}\right) + i\cdot\sin\left(-\frac{5\pi}{6}\right)\right)\right]^{9} =$$

$$= \left(\frac{1}{4}\right)^{9}\left(\cos\left(-\frac{15\pi}{2}\right) + i\cdot\sin\left(-\frac{15\pi}{2}\right)\right) =$$

$$= \frac{1}{4^9}\left(\cos\left(\frac{\pi}{2}\right) + i \cdot \sin\left(\frac{\pi}{2}\right)\right) =$$

$$= \frac{1}{4^9}\left(0 + i\right) = \frac{1}{4^9} i = \frac{1}{262\,144}\, i.$$

f) $(2+2i)^6 = \left[2\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i \cdot \sin\left(\frac{\pi}{4}\right)\right)\right]^6 =$

$$= 2^6 \cdot \sqrt{2}^6\left(\cos\left(\frac{3\pi}{2}\right) + i \sin\left(\frac{3\pi}{2}\right)\right) =$$

$$= 64 \cdot 8\,(0-1) = -512.$$

8.

a) $\sqrt[4]{-16} = \sqrt[4]{16\left(\cos(\pi) + i \cdot \sin(\pi)\right)} =$

$$= \sqrt[4]{16}\left(\cos\left(\frac{\pi}{4} + k \cdot \frac{2\pi}{4}\right) + i \cdot \sin\left(\frac{\pi}{4} + k \cdot \frac{2\pi}{4}\right)\right),$$

ahol $k = 0, 1, 2, 3.$

$$2\left(\cos\left(\frac{\pi}{4}+k\cdot\frac{\pi}{2}\right)+i\cdot\sin\left(\frac{\pi}{4}+k\cdot\frac{\pi}{2}\right)\right),$$

ahol $k=0,1,2,3$.

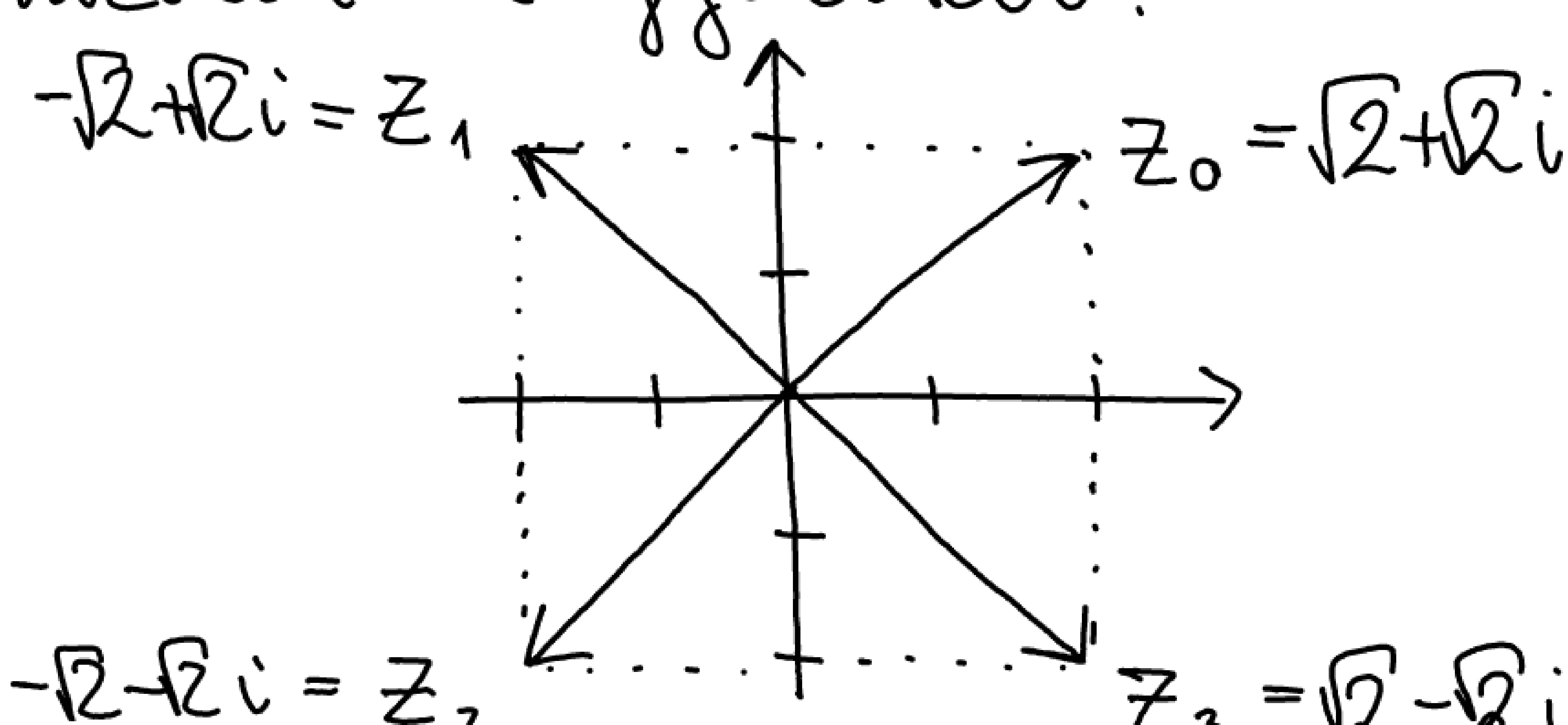
$$z_0=2\left(\cos\left(\frac{\pi}{4}\right)+i\cdot\sin\left(\frac{\pi}{4}\right)\right)=\sqrt{2}+\sqrt{2}i$$

$$z_1=2\left(\cos\left(\frac{3\pi}{4}\right)+i\cdot\sin\left(\frac{3\pi}{4}\right)\right)=-\sqrt{2}+\sqrt{2}i$$

$$z_2=2\left(\cos\left(\frac{5\pi}{4}\right)+i\cdot\sin\left(\frac{5\pi}{4}\right)\right)=-\sqrt{2}-\sqrt{2}i$$

$$z_3=2\left(\cos\left(\frac{7\pi}{4}\right)+i\cdot\sin\left(\frac{7\pi}{4}\right)\right)=\sqrt{2}-\sqrt{2}i$$

Ábrázolva a gyököket.



b

$$\sqrt{2i} = \sqrt{2\left(\cos\left(\frac{\pi}{2}\right) + i\cdot\sin\left(\frac{\pi}{2}\right)\right)} =$$

$$= \sqrt{2}\left(\cos\left(\frac{\pi}{4} + k\cdot\frac{2\pi}{2}\right) + i\cdot\sin\left(\frac{\pi}{4} + k\cdot\frac{2\pi}{2}\right)\right) =$$

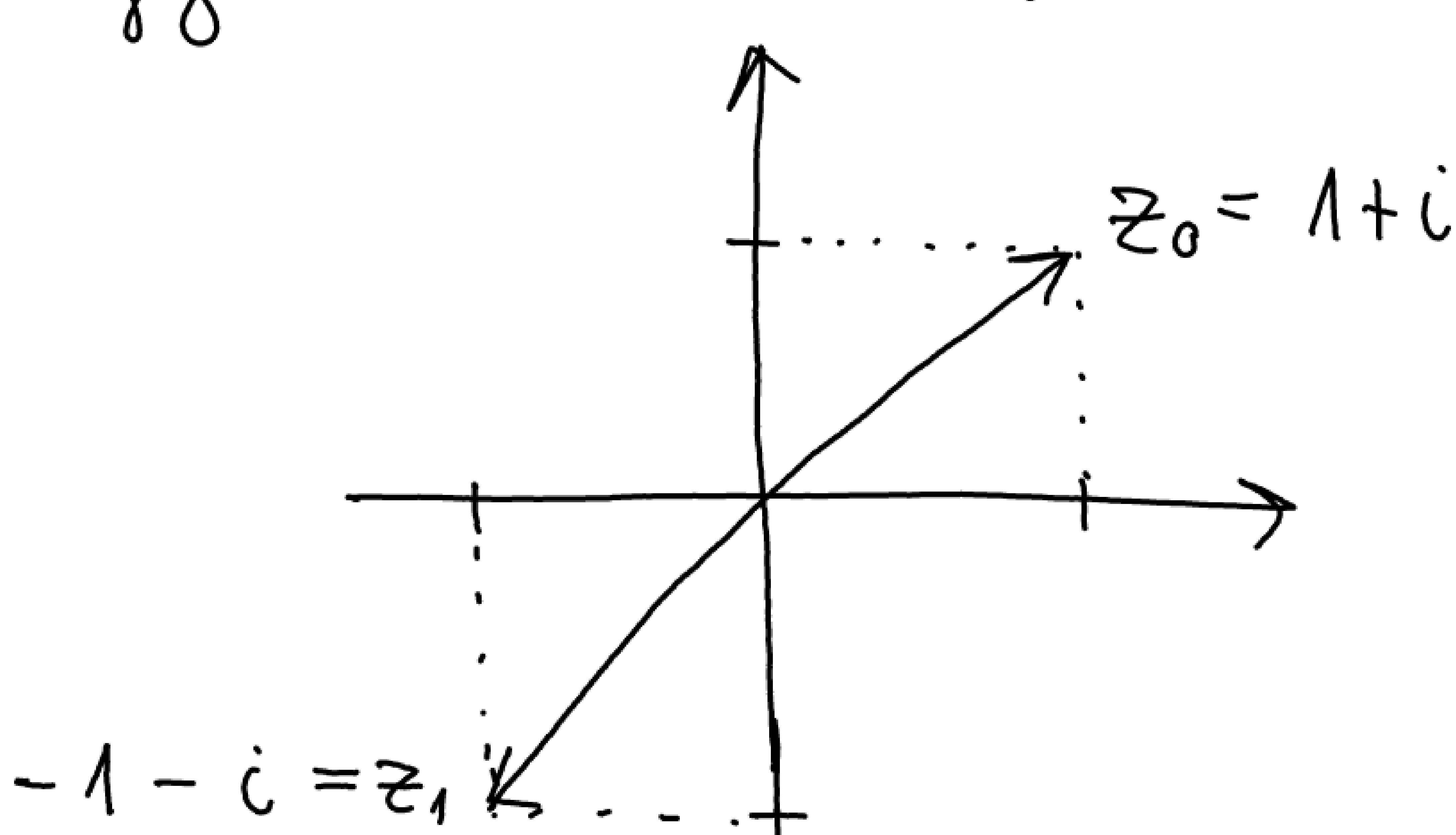
$$= \sqrt{2}\left(\cos\left(\frac{\pi}{4} + k\pi\right) + i\cdot\sin\left(\frac{\pi}{4} + k\pi\right)\right),$$

ahol $k = 0, 1$.

$$z_0 = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\cdot\sin\left(\frac{\pi}{4}\right)\right) = 1 + i$$

$$z_1 = \sqrt{2}\left(\cos\left(\frac{5\pi}{2}\right) + i\cdot\sin\left(\frac{5\pi}{2}\right)\right) = -1 - i$$

A gyökök ábrázolva.



c)

$$\sqrt[5]{-243i} = \sqrt[5]{243\left(\cos\left(\frac{3\pi}{2}\right) + i\cdot\sin\left(\frac{3\pi}{2}\right)\right)} =$$

$$= \sqrt[5]{243}\left(\cos\left(\frac{3\pi}{10} + k\cdot\frac{2\pi}{5}\right) + i\cdot\sin\left(\frac{3\pi}{10} + k\cdot\frac{2\pi}{5}\right)\right) =$$

$$= 3\left(\cos\left(\frac{3\pi}{10} + k\cdot\frac{2\pi}{5}\right) + i\cdot\sin\left(\frac{3\pi}{10} + k\cdot\frac{2\pi}{5}\right)\right),$$

ahol $k = 0, 1, 2, 3, 4$.

d)

$$\sqrt[3]{-4\sqrt{2} + 4\sqrt{2}\,i} = \sqrt[3]{8\left(\cos\left(\frac{3\pi}{4}\right) + i\cdot\sin\left(\frac{3\pi}{4}\right)\right)} =$$

$$= \sqrt[3]{8}\left(\cos\left(\frac{\pi}{4} + k\cdot\frac{2\pi}{3}\right) + i\cdot\sin\left(\frac{\pi}{4} + k\cdot\frac{\pi}{3}\right)\right) =$$

$$= 2\left(\cos\left(\frac{\pi}{4} + k\cdot\frac{2\pi}{3}\right) + i\cdot\sin\left(\frac{\pi}{4} + k\cdot\frac{\pi}{3}\right)\right),$$

ahol $k = 0, 1, 2$.

e) $\sqrt[3]{-\sqrt{3}+i} = \sqrt[3]{2\left(\cos\left(\frac{3\pi}{4}\right)+i\cdot\sin\left(\frac{3\pi}{4}\right)\right)} =$

$= \sqrt[3]{2}\left(\cos\left(\frac{\pi}{4}+k\cdot\frac{2\pi}{3}\right)+i\cdot\sin\left(\frac{\pi}{4}+k\cdot\frac{2\pi}{3}\right)\right),$

ahol $k = 0, 1, 2$.

f) $\sqrt[4]{1+\sqrt{3}i} = 2\sqrt[4]{\cos\left(\frac{\pi}{3}\right)+i\cdot\sin\left(\frac{\pi}{3}\right)} =$

$= \sqrt[4]{2}\left(\cos\left(\frac{\pi}{12}+k\cdot\frac{\pi}{2}\right)+i\cdot\sin\left(\frac{\pi}{12}+k\cdot\frac{\pi}{2}\right)\right),$

ahol $k = 0, 1, 2, 3$.

9.

a) $z^3-1=0 \iff z=\sqrt[3]{1}=\sqrt[3]{\cos(0)+i\cdot\sin(0)}$

$= \cos\left(k\cdot\frac{2\pi}{3}\right)+i\cdot\sin\left(k\cdot\frac{2\pi}{3}\right) \quad (k=0,1,2)$

$z_0 = 1, \quad z_1 = -\frac{1}{2}+\frac{\sqrt{3}}{2}i, \quad z_2 = -\frac{1}{2}-\frac{\sqrt{3}}{2}i.$

b) $z^3 = 1+i \Leftrightarrow z = \sqrt[3]{1+i} = \sqrt[3]{\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)}$

$= \sqrt[6]{2}\left(\cos\left(\frac{\pi}{12} + k\cdot\frac{2\pi}{3}\right) + i\cdot\sin\left(\frac{\pi}{12} + k\cdot\frac{2\pi}{3}\right)\right),$

ahol $k = 0, 1, 2$.

c) $z^2 - 8z + 25 = 0$

$$(z-4)^2 - 16 + 25 = 0$$

$$(z-4)^2 = -9$$

$$z - 4 = \sqrt{-9}$$

Továbbá nyilván $\sqrt{-9} = 3i, -3i$, amiből a megoldások

$$z_1 = 4+3i \text{ és } z_2 = 4-3i.$$

A polinom szorzattá alakítva

$$z^2 - 8z + 25 = (z-4-3i)(z-4+3i).$$

(d*) $z^2+(2-2i)z+2i=0$

$$\left(z+(1-i)\right)^2-(1-i)^2+2i=0$$

$$\left(z+(1-i)\right)^2-1+2i-i^2+2i=0$$

$$\left(z+(1-i)\right)^2+4i=0$$

$$z+(1-i)=\sqrt{-4i}$$

Mivel $\sqrt{-4i}=\sqrt{4\cos\left(\frac{3\pi}{2}\right)+i\cdot\sin\left(\frac{3\pi}{2}\right)}=$

$$=2\left(\cos\left(\frac{3\pi}{4}+k\cdot 2\pi\right)+i\cdot\sin\left(\frac{3\pi}{4}+k\cdot 2\pi\right)\right)\ (k=0,1)$$

$$\sqrt{-4i}=-\sqrt{2}+\sqrt{2}i\ ,\ \sqrt{2}-\sqrt{2}i,$$

ezért

$$z_1+(1-i)=-\sqrt{2}+\sqrt{2}i \Leftrightarrow z_1=-\sqrt{2}-1+(\sqrt{2}+1)i$$

$$z_2+(1-i)=\sqrt{2}-\sqrt{2}i \Leftrightarrow z_2=\sqrt{2}-1-(\sqrt{2}-1)i$$

e* $z^6+2z^3+2=0$

$$(z^3+1)^2-1+2=0$$

$$(z^3+1)^2=-1$$

$$z^3-1=\sqrt{-1}\,,$$

továbbá tudjuk, hogy $\sqrt{-1}=i,-i$, ezért adódik, hogy

$$z^3=1+i \quad \text{vagy} \quad z^3=1-i$$

1. eset:

$$z_1^3=1+i \Leftrightarrow z_1=\sqrt[3]{1+i}=\sqrt[3]{\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right)+i\cdot\sin\left(\frac{\pi}{4}\right)\right)}$$

$$=\sqrt[6]{2}\left(\cos\left(\frac{\pi}{12}+k\cdot\frac{2\pi}{3}\right)+i\cdot\sin\left(\frac{\pi}{12}+k\cdot\frac{2\pi}{3}\right)\right),$$

ahol $k=0,1,2.$

2. eset:

$$z_2^3 = 1 - i \Leftrightarrow z_2 = \sqrt[3]{1-i} = \sqrt[3]{\sqrt{2}\cos\left(\frac{7\pi}{4}\right) + i \cdot \sin\left(\frac{7\pi}{4}\right)} =$$

$$= \sqrt[6]{2}\left(\cos\left(\frac{7\pi}{12} + l \cdot \frac{2\pi}{3}\right) + i \cdot \sin\left(\frac{7\pi}{12} + l \cdot \frac{2\pi}{3}\right)\right),$$

ahol $l = 0, 1, 2$.