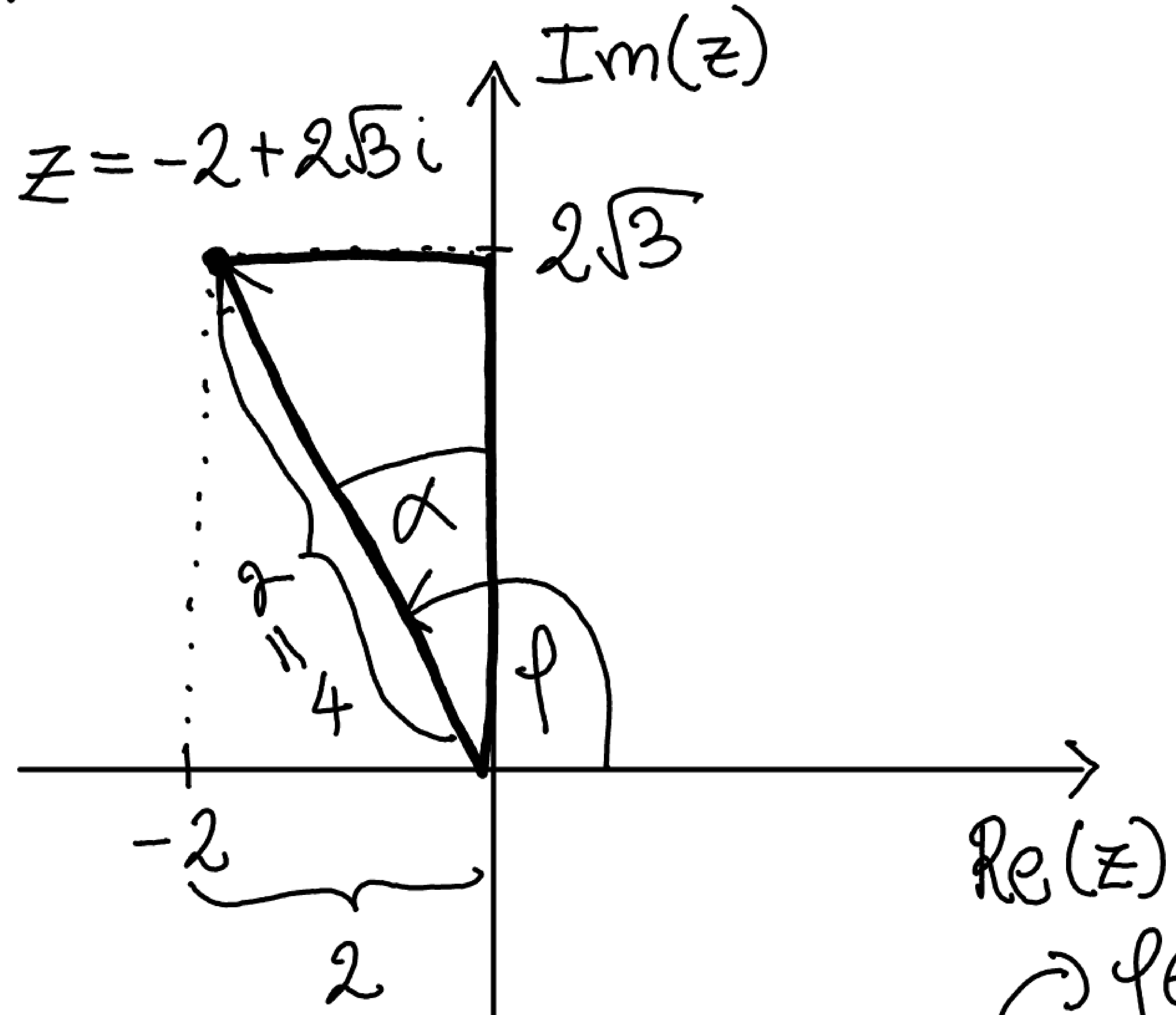
1. feladat

$$z=-2+2\sqrt{3}\,i$$



$$z=r\left(\cos(\varphi)+i\sin(\varphi)\right) \qquad \varphi\in[0,2\pi)$$

$$r=\sqrt{(-2)^2+(2\sqrt{3})^2}=\sqrt{4+4\cdot 3}$$

$$=\sqrt{16}=4$$

$$\sin(\alpha)=\frac{2}{4}=\frac{1}{2}\iff \alpha=30^\circ=\frac{\pi}{6}.$$

$$\underbrace{\varphi=\frac{\pi}{2}+\alpha=\frac{\pi}{2}+\frac{\pi}{6}=\frac{3\pi+\pi}{6}=\frac{4\pi}{6}}$$

$$z=4\left(\cos\left(\frac{4\pi}{6}\right)+i\sin\left(\frac{4\pi}{6}\right)\right)$$

$$z^4=4^4\left(\cos\left(4\cdot\frac{4\pi}{6}\right)+i\sin\left(4\cdot\frac{4\pi}{6}\right)\right)$$

$$=4^4\left(\cos\left(\frac{16\pi}{6}\right)+i\cdot\sin\left(\frac{16\pi}{6}\right)\right)$$

$$=4^4\left(\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{6}\right)\right)$$

$$\frac{16\pi}{6}=2\pi+\frac{4\pi}{6}=2\pi+\frac{2\pi}{3}$$

$$z = 4\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$$

$$\sqrt[4]{z} = \sqrt[4]{4}\left(\cos\left(\frac{1}{4}\cdot\frac{2\pi}{3} + k\cdot\frac{2\pi}{4}\right) + i\cdot\sin\left(\frac{1}{4}\cdot\frac{2\pi}{3} + k\cdot\frac{2\pi}{4}\right)\right),$$

ahol $k = 0, 1, 2, 3$.

$$\sqrt[4]{z} = \sqrt[4]{4}\left(\cos\left(\frac{\pi}{6} + k\cdot\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{6} + k\cdot\frac{\pi}{2}\right)\right),$$

ahol $k = 0, 1, 2, 3$.

$$\left.\begin{array}{l} \sqrt[n]{z} = \sqrt[n]{r}\left(\cos\left(\frac{\varphi}{n} + k\cdot\frac{2\pi}{n}\right) + i\sin\left(\frac{\varphi}{n} + k\cdot\frac{2\pi}{n}\right)\right), \\ \text{ahol } k = 0, 1, \dots, n-1 \end{array}\right\} \text{általánosan}$$

Megj:

$$z = -2 + 2\sqrt{3}i$$

$$z^2 = (-2+2\sqrt{3}i)^2 = (-2)^2 + 2\cdot(-2)\cdot 2\sqrt{3}i + (2\sqrt{3}i)^2$$

$$= 4 - 8\sqrt{3}i + \underbrace{4\cdot 3}_{12}\cdot\underbrace{i^2}_{-1}$$

$$= 4 - 8\sqrt{3}i - 12$$

$$= -8 - 8\sqrt{3}i$$

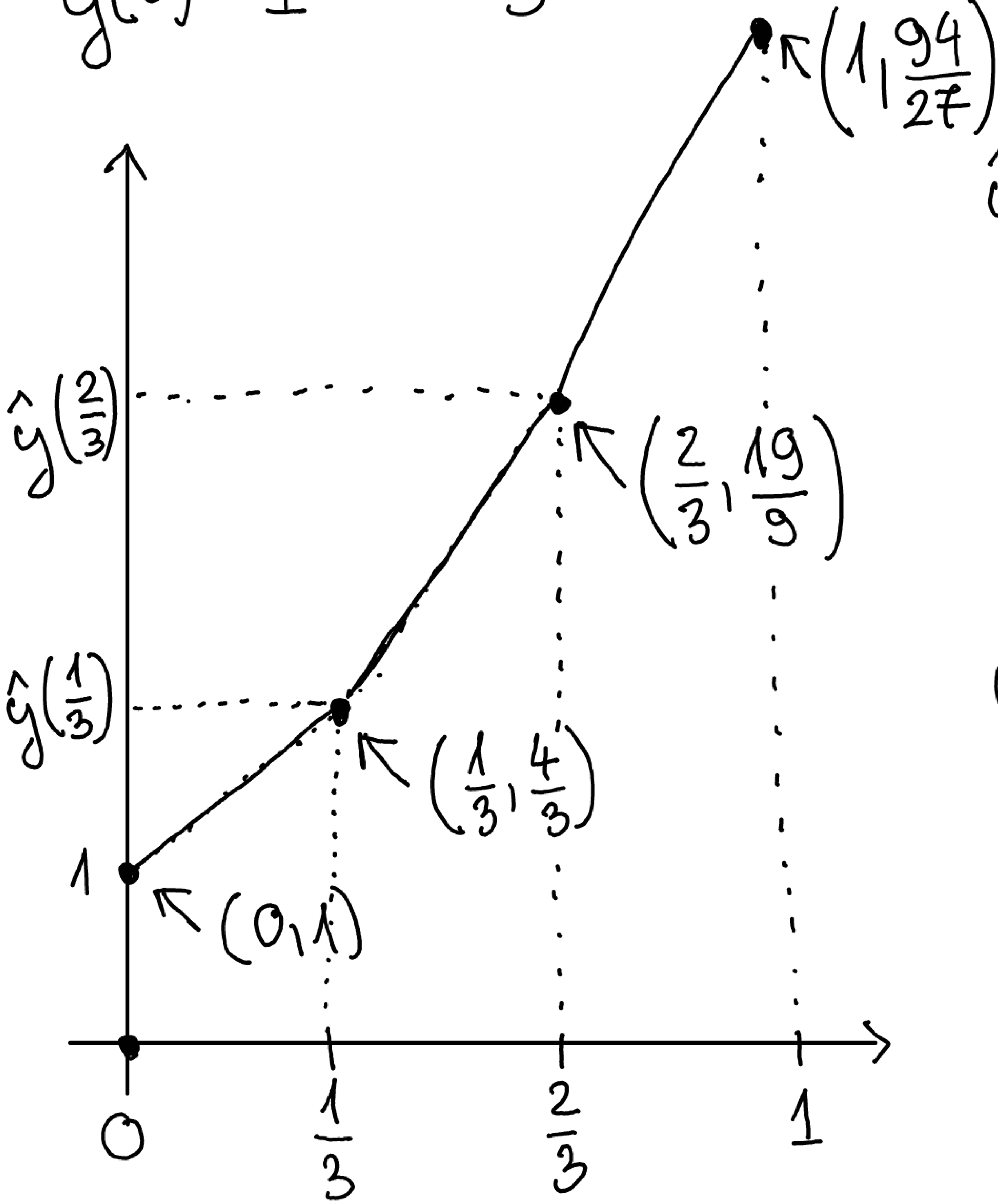
$$z^4 = (z^2)^2 = (-8-8\sqrt{3}i)^2 = 64 + 128\sqrt{3}i + 64\cdot 3\,\underbrace{i^2}_{-1}$$

$$= 64 + 128\sqrt{3}i + 192\cdot(-1)$$

$$= -128 + 128\sqrt{3}i$$

2. feladat

$$\left.\begin{aligned} y' &= 3x + y \\ y(0) &= 1 \end{aligned}\right\} \quad y(1) \approx ?$$



$$\hat{y}\left(\frac{1}{3}\right) = \underbrace{y(0)}_{1} + (3 \cdot 0 + 1) \cdot \frac{1}{3}$$

$$= 1 + 1 \cdot \frac{1}{3} = \frac{4}{3}$$

$$\hat{y}\left(\frac{2}{3}\right) = \underbrace{\hat{y}\left(\frac{1}{3}\right)}_{\frac{4}{3}} + \underbrace{\left(3 \cdot \frac{1}{3} + \frac{4}{3}\right)}_{\frac{7}{3}} \cdot \frac{1}{3}$$

$$= \frac{4}{3} + \frac{7}{9} = \frac{12+7}{9} = \frac{19}{9}$$

$$\hat{y}(1) = \underbrace{\hat{y}\left(\frac{2}{3}\right)}_{\frac{19}{9}} + \underbrace{\left(3 \cdot \frac{2}{3} + \frac{19}{9}\right)}_{\frac{37}{9}} \cdot \frac{1}{3}$$

$$= \frac{19}{9} + \frac{37}{27} = \frac{57+37}{27} = \frac{94}{27}$$

$$y(1) \approx \frac{94}{27}.$$

3. feladat

$$\left.\begin{aligned} xy' &= y + x + 1 \\ y(1) &= 1 \end{aligned}\right\}$$

$$xy' = y + x + 1$$

$$y' = \frac{1}{x}y + \underbrace{1 + \frac{1}{x}}_{\neq 0 \to \text{inhomogén}}$$

$$\underbrace{y' = p(x)y + q(x)}_{\text{elsőrendű lineáris}}$$

A homogén egyenlet

$$y' = \frac{1}{x}y \qquad \underbrace{y' = f(x)g(y)}_{\text{szeparálható}}$$

$$\frac{1}{y}\cdot y' = \frac{1}{x}$$

$$\int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$$

$$\ln|y| = \ln|x| + c$$

$$e^{\ln|y|} = e^{\ln|x| + c} = e^{\ln|x|}\cdot \underbrace{e^{c}}_{C>0}$$

$$|y| = \underbrace{C}_{>0}\cdot|x|$$

$$y = \underbrace{C}_{\in\mathbb{R}}\cdot x$$

A homogén egyenlet ált. megoldása:

$$y_H = C\cdot x$$

Keressünk egy partikuláris megoldást az inhomogén egyenlethez

$$\left.\begin{aligned} y_p &= c(x)\cdot x \\ y_p' &= c'(x)x + c(x) \end{aligned}\right\}$$

Helyettesítsük be:

$$\underbrace{y_p' = \frac{1}{x}y_p + 1 + \frac{1}{x}}$$

$$c'(x)x + \cancel{c(x)} = \cancel{\frac{1}{x}c(x)\cdot x} + 1 + \frac{1}{x}$$

$$c'(x)x = 1 + \frac{1}{x}$$

$$c'(x) = \frac{1}{x} + \frac{1}{x^2}$$

$$\Rightarrow c(x) = \int \frac{1}{x} + \frac{1}{x^2}\,dx \qquad \frac{1}{x^2} = x^{-2}$$

$$= \ln|x| - x^{-1}$$

$$= \ln|x| - \frac{1}{x}$$

Vagyis

$$y_p = x\cdot c(x)$$

$$y_p = x\left(\ln|x| - \frac{1}{x}\right)$$

$$y_p = x\ln|x| - 1$$

Az inhomogén általános megoldása:

$$y = y_p + y_H$$

$$y = x\ln|x| - 1 + c\cdot x$$

$$\underbrace{y = x\ln|x| - 1 + c\cdot x}_{}$$

$y(1) = 1$ miatt:

$$1 = y(1) = 1\cdot\underbrace{\ln|1|}_{0} - 1 + c\cdot 1$$

$$1 = 1\cdot 0 - 1 + c$$

$$1 = c - 1$$

$$\underbrace{c = 2}_{}$$

A feladat megoldása:

$$y = x\ln|x| - 1 + 2x$$

4. feladat

$$y'' - 4y' + 8y = \underbrace{e^{2x}\cos(x)}_{\neq 0 \to \text{ inhomogén}}$$

$$\underbrace{py'' + qy' + ry = f(x)}_{\text{állandó együtthatós másodrendű}}$$

A homogén egyenlet

$$y'' - 4y' + 8y = 0$$

A karakterisztikus egyenlet

$$\lambda^2 - 4\lambda + 8 = 0$$

$$(\lambda - 2)^2 + 8 - 4 = 0$$

$$(\lambda - 2)^2 = -4 \to$$

$$\lambda - 2 = \sqrt{-4}$$

$$\lambda - 2 = \pm 2i$$

$$\lambda = 2 \pm 2i$$

A homogén általános megoldása:

$$y_H = c_1 \cdot e^{2x}\cos(2x) + c_2 e^{2x}\sin(2x)$$

__Megj__: (i) Ha pl. a gyökök $\lambda_1 = 2$ és $\lambda_2 = -1$.

$$y_H = c_1 e^{2x} + c_2 e^{-x}$$

(ii) Ha pl a gyök $\lambda = -2$

$$y_H = c_1 e^{-2x} + c_2 x e^{-2x}$$

Keressünk egy megoldást az inhomogén egyenletben

$$y''-4y'+8y=\underbrace{e^{2x}\cos(x)}_{f(x)}$$

$$f(x)=P_n(x)e^{\alpha x}\cos(\beta x)$$

vagy

$$f(x)=P_n(x)e^{\alpha x}\sin(\beta x)$$

Most:

$$e^{2x}\cos(x)=\underbrace{1}_{P_0(x)}\cdot\underbrace{e^{2x}}_{\underbrace{e^{\alpha x}}_{\alpha=2}}\underbrace{\cos(x)}_{\underbrace{\cos(\beta x)}_{\beta=1}}$$

Gyöke-e a kar. egy.-nek $\alpha\pm\beta i=2\pm i$?

NEM $\longrightarrow$

Az alak, amiben keressük a megoldást: y_p

$$=Q_0(x)e^{\alpha x}\cos(\beta x)+R_0(x)e^{\alpha x}\sin(\beta x)$$

$$y_p=q_0e^{2x}\cos(x)+r_0e^{2x}\sin(x)$$

$$y_p'=q_02e^{2x}\cos(x)-q_0e^{2x}\sin(x)+r_02e^{2x}\sin(x)+r_0e^{2x}\cos(x)$$

$$y_p''=q_04e^{2x}\cos(x)-q_02e^{2x}\sin(x)-q_02e^{2x}\sin(x)-q_0e^{2x}\cos(x)+r_04e^{2x}\sin(x)+r_02e^{2x}\cos(x)+r_02e^{2x}\cos(x)-r_0e^{2x}\sin(x)$$

$$\left.\begin{aligned}
y_p &= \cancel{q_0 e^{2x}\cos(x)} + \cancel{r_0 e^{2x}\sin(x)} \\
y_p' &= \cancel{q_0 2 e^{2x}\cos(x)} - \cancel{q_0 e^{2x}\sin(x)} \\
&\quad + \cancel{r_0 2 e^{2x}\sin(x)} + \cancel{r_0 e^{2x}\cos(x)} \\
y_p'' &= \cancel{q_0 4 e^{2x}\cos(x)} - \cancel{q_0 2 e^{2x}\sin(x)} \\
&\quad - \cancel{q_0 2 e^{2x}\sin(x)} - \cancel{q_0 e^{2x}\cos(x)} \\
&\quad + \cancel{r_0 4 e^{2x}\sin(x)} + r_0 2 e^{2x}\cos(x) \\
&\quad + \cancel{r_0 2 e^{2x}\cos(x)} - \cancel{r_0 e^{2x}\sin(x)}
\end{aligned}\right\} \quad y_p'' - 4y_p' + 8y_p = e^{2x}\cos(x)$$

$$\left.\begin{aligned}
e^{2x}\cos(x) &= 3q_0 e^{2x}\cos(x) + 3r_0 e^{2x}\sin(x) \\
\cos(x) &= 3q_0\cos(x) + 3r_0\sin(x)
\end{aligned}\right\} \quad \text{Egy jó választás: } q_0 = \frac{1}{3} \text{ és } r_0 = 0.$$

Egy jó megoldása az inhomogén egyenletnek:

$$y_p = \underbrace{q_0}_{\frac{1}{3}} e^{2x}\cos(x) + \underbrace{r_0}_{0} e^{2x}\sin(x)$$

$$y_p = \frac{1}{3} e^{2x}\cos(x)$$

<u>Az inhomogén általános megoldása</u>

$$y = y_p + y_H$$

$$y = \frac{1}{3} e^{2x}\cos(x) + c_1 e^{2x}\cos(2x) + c_2 e^{2x}\sin(2x).$$

Megj: Pl ha $\alpha = 2,\ \beta = 2 \Rightarrow 2 \pm 2i$ komplex gyöke:

$$\Rightarrow y_p = x\left(Q_0(x)e^{\alpha x}\cos(\beta x) + R_0(x)e^{\alpha x}\sin(\beta x)\right)$$

$$= q_0 \cdot x e^{2x}\cos(2x) + r_0 x e^{2x}\sin(2x)$$

Pl: A kar. egy. gyökei: $\lambda_1 = 2,\ \lambda_2 = -1$

és $f(x) = \underbrace{(x+2)}_{P_1(x)}\underbrace{e^{2x}}_{e^{\alpha x}} \cdot \underbrace{\cos(0 \cdot x)}_{\cos(\beta x)}$ $\Big\}$ $\alpha \pm \beta i = 2 \rightarrow$ 1-szeres valós gyök

$$\Rightarrow y_p = x\left(Q_1(x)e^{\alpha x}\cos(\beta x) + R_1(x)e^{\alpha x}\sin(\beta x)\right)$$

$$= x(q_1 x + q_0)e^{2x}\underbrace{\cos(0 \cdot x)}_{1} + \underbrace{x(r_1 x + r_0)e^{2x}\sin(0 \cdot x)}_{0}$$

$$= x(q_1 x + q_0)e^{2x}$$

Pl: A kar. egy. gyöke: $\lambda = -1$

$$f(x) = \underbrace{(x^2+2x+1)}_{P_2(x)}\underbrace{e^{-x}}_{e^{\alpha x}} \qquad \left.\beta = 0\right\} \; -1 \pm 0i = -1 \Rightarrow$$ kétszeres valós gyök.

$$\Rightarrow y_p = x^2 Q_2(x) e^{-\alpha x} = x^2 (q_2 x^2 + q_1 x + q_0) e^{-x}$$