(1.) $A = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$

$$p(\lambda) = \begin{vmatrix} 7-\lambda & 3 \\ 3 & -1-\lambda \end{vmatrix} = 0$$

$$(7-\lambda)(-1-\lambda) - 3\cdot 3 = 0$$

$$\lambda^2 - 6\lambda - 16 = 0$$

$$(\lambda-3)^2 - 16 - 9 = 0$$

$$(\lambda-3)^2 = 25$$

$$\lambda - 3 = \pm 5$$

$$\lambda = 3 \pm 5$$

A sajátértékek:

$$\lambda_1 = 3-5 = -2$$

$$\lambda_2 = 3+5 = 8$$

A sajátvektorok:

- $\lambda_1 = -2$:

$$\underbrace{\underbrace{\begin{bmatrix} 7-(-2) & 3 \\ 3 & -1-(-2) \end{bmatrix}}_{(A-(-2)I)}}_{\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}} \cdot \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\underline{v}} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\underline{0}}$$

$$\left[\begin{array}{cc|c} 9 & 3 & 0 \\ 3 & 1 & 0 \end{array}\right] \overset{\frac{1}{3}S_1}{\rightsquigarrow} \left[\begin{array}{cc|c} 3 & 1 & 0 \\ 3 & 1 & 0 \end{array}\right]$$

$$\overset{S_2 - S_1}{\rightsquigarrow} \left[\begin{array}{cc|c} 3 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \overset{\frac{1}{3}S_1}{\rightsquigarrow} \left[\begin{array}{cc|c} 1 & \frac{1}{3} & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{array}{r} x + \frac{1}{3}y = 0 \\ y = t \end{array}\right\} \quad \begin{array}{l} x = -\frac{1}{3}t \\ y = t \end{array}$$

Tehát

$$\underline{v} = t \cdot \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$$

A független sajátvektorok

$\lambda_1 = -2$-höz:

$$3 \cdot \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \underline{v}_1$$

- $\lambda_2 = 8$:

Mivel

$$\underbrace{A = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} \rightsquigarrow A^T = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}}_{A = A^T}$$

szimmetrikus.

Ezért $\lambda_2 = 8$-hoz jó sajátvektor:

$$\underline{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix},$$

mert $\underline{v}_1 \perp \underline{v}_2$.

Összefoglalva:

$$\underline{v}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \lambda_1 = -2$$

és

$$\underline{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \lambda_2 = 8$$

Kell: $A = C \cdot D \cdot C^{-1}$

$$C = \begin{bmatrix} -1 & 3 \\ 3 & 1 \end{bmatrix} \quad \text{és} \quad D = \begin{bmatrix} -2 & 0 \\ 0 & 8 \end{bmatrix}$$

<u>Megj</u>:

normáljuk le $\underline{v}_1$-et és $\underline{v}_2$-t:

$$\left.\begin{aligned} \frac{\underline{v}_1}{|\underline{v}_1|} &= \frac{1}{\sqrt{10}} \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \ \lambda_1 = -2 \\ \frac{\underline{v}_2}{|\underline{v}_2|} &= \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \ \lambda_2 = 8 \end{aligned}\right\}$$

Ezért, ha

$$C = \frac{1}{\sqrt{10}} \begin{bmatrix} -1 & 3 \\ 3 & 1 \end{bmatrix} \text{ és } D = \begin{bmatrix} -2 & 0 \\ 0 & 8 \end{bmatrix},$$

akkor

$$C^{-1} = C^{T} = \frac{1}{\sqrt{10}} \begin{bmatrix} -1 & 3 \\ 3 & 1 \end{bmatrix}.$$

$$A^5 = (C \cdot D \cdot C^{-1})^5 = C \cdot D^5 C^{-1}$$

$$= C \cdot \underbrace{\begin{bmatrix} (-2)^5 & 0 \\ 0 & 8^5 \end{bmatrix}}_{D^5} \cdot C^{-1}$$

2. $f(x,y) = x^2 \sin(\pi y) + \ln(xy)$

$P(1,1)$

$\underline{v}\,(3,4)$

$$f'_{\underline{v}}(1,1) = \left\langle \nabla f(1,1) \,\middle|\, \frac{\underline{v}}{|\underline{v}|} \right\rangle$$

$$f'_x(x,y) = 2x \cdot \sin(\pi y) + \frac{1}{xy} \cdot y$$

$$f'_y(x,y) = x^2 \cdot \cos(\pi y) \cdot \pi + \frac{1}{xy} \cdot x$$

$$f'_x(1,1) = 1$$

$$f'_y(1,1) = 1 - \pi$$

A gradiens az $(1,1)$ pontban:

$$\nabla f(1,1) = (1,\ 1-\pi)$$

Az irány normáltja:

$$\frac{\underline{v}}{|\underline{v}|} = \frac{1}{\sqrt{3^2+4^2}}(3,4) = \frac{1}{5}(3,4) = \left(\frac{3}{5}, \frac{4}{5}\right)$$

A vége:

$$f'_{\underline{v}}(1,1) = 1 \cdot \frac{3}{5} + (1-\pi) \cdot \frac{4}{5} = \frac{7}{5} - \frac{4\pi}{5}.$$

Megj: Érintősík az (1,1) pontban:

A normálvektor:

$\underline{n}(1, 1-\pi, -1)$

Egy pont a síkon:

$$R(1, 1, f(1,1)) = (1,1,0)$$

$$f(1,1) = 1^2 \cdot \underbrace{\sin(\pi)}_{=0} + \underbrace{\ln(1\cdot 1)}_{0} = 0$$

A sík egyenlete:

$$0 = 1\cdot(x-1) + (1-\pi)\cdot(y-1) - 1\cdot(z-0).$$

3. $f(x,y) = x^3 - 12x + y^2 + 2$

$$\left.\begin{aligned} f'_x(x,y) &= 3x^2 - 12 = 0 \quad (1) \\ f'_y(x,y) &= 2y = 0 \quad (2) \end{aligned}\right\}$$

(1) $3x^2 - 12 = 0$

$x^2 - 4 = 0$

$x^2 = 4$

$x = \pm 2$

(2) $2y = 0$

$y = 0$

A kritikus pontok:

$(-2,0)$, $(2,0)$.

$$\left.\begin{aligned} f''_{xx}(x,y) &= 6x \\ f''_{yy}(x,y) &= 2 \\ f''_{xy}(x,y) &= f''_{yx}(x,y) = 0 \end{aligned}\right\}$$

$$H = \begin{bmatrix} 6x & 0 \\ 0 & 2 \end{bmatrix}$$

A $(-2,0)$-ban:

$$H = \begin{bmatrix} -12 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow \det(H) = (-12)\cdot 2 - 0\cdot 0 = -24 < 0$$

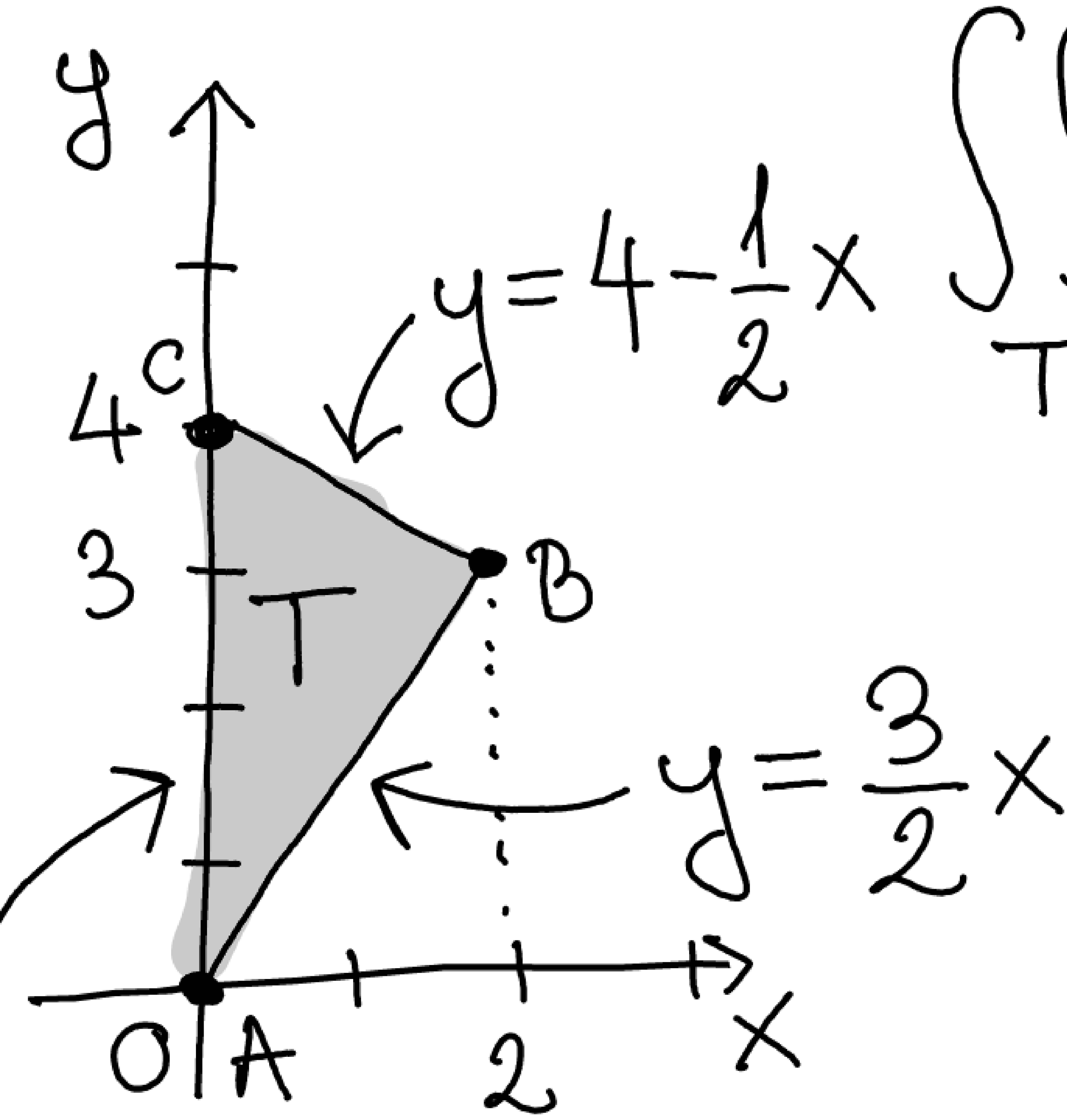
$\Rightarrow (-2,0)$ nyeregpont

A $(2,0)$-ban:

$$H = \begin{bmatrix} 12 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow \begin{array}{l} \det(H) = 24 > 0 \\ \text{és} \\ f''_{xx}(2,0) = 12 > 0 \end{array}$$

$\Rightarrow (2,0)$ lokális minimum

4. $f(x,y) = e^{3x-2y}$



$$\iint_T e^{3x-2y}\,dA = \int_0^2 \int_{\frac{3}{2}x}^{4-\frac{1}{2}x} e^{3x-2y}\,dy\,dx$$

$$T = \left\{ \begin{array}{l} 0 \leqq x \leqq 2, \\ \frac{3}{2}x \leqq y \leqq 4 - \frac{1}{2}x \end{array} \right\}$$

$$= \int_0^2 \left[e^{3x-2y} \cdot \frac{-1}{2} \right]_{y=\frac{3}{2}x}^{y=4-\frac{1}{2}x} dx$$

$$= \int_0^2 -\frac{1}{2} e^{3x-2\left(4-\frac{1}{2}x\right)} + \frac{1}{2} e^{3x-2\left(\frac{3}{2}x\right)}\,dx$$

$$= \int_0^2 -\frac{1}{2} e^{4x-8} + \frac{1}{2} \underbrace{e^0}_{1}\,dx$$

$$= \int_0^2 -\frac{1}{2}e^{4x-8} + \frac{1}{2}\,dx$$

$$= \left[-\frac{1}{2}e^{4x-8}\cdot\frac{1}{4} + \frac{1}{2}x \right]_{x=0}^{x=2}$$

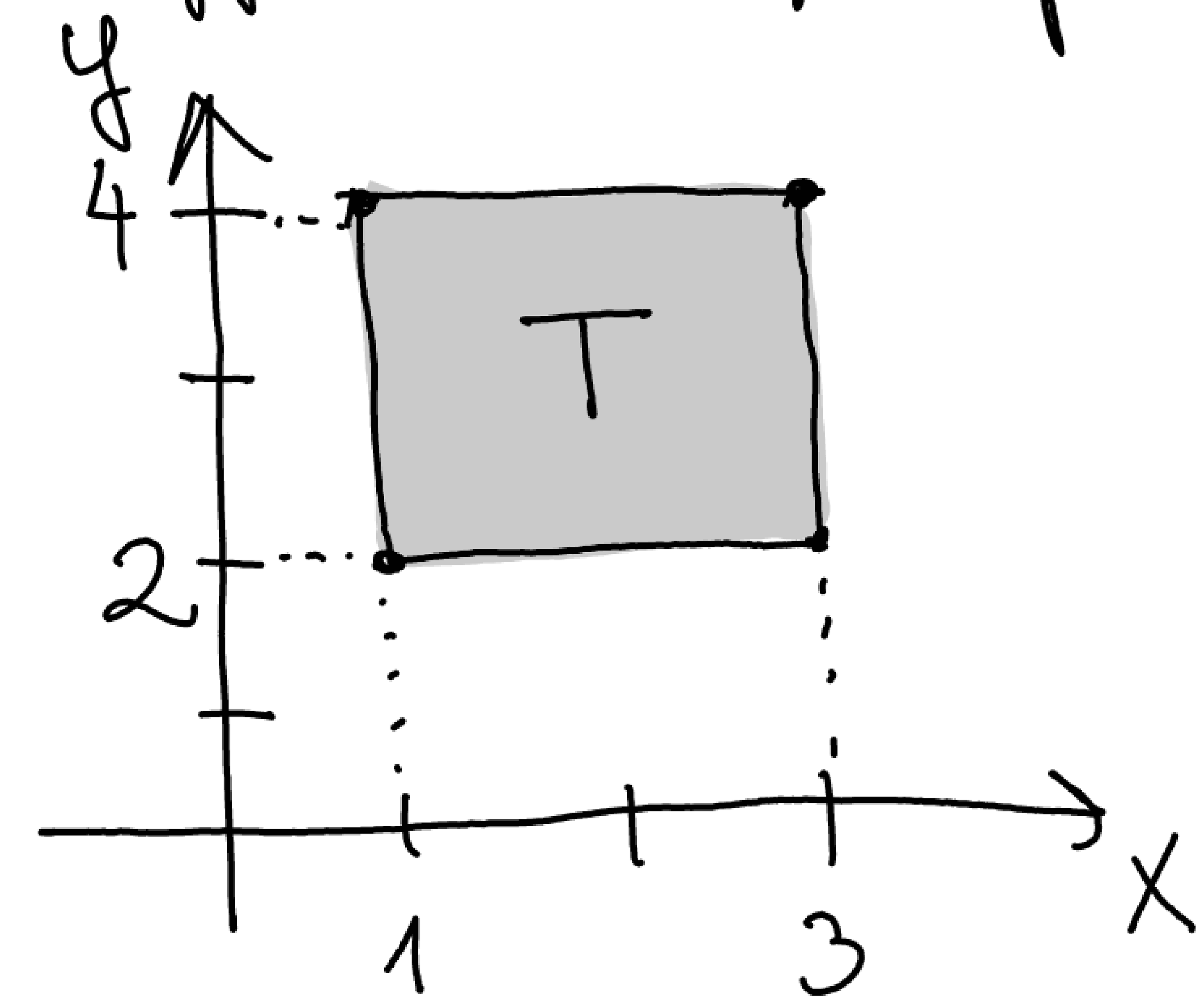
$$= -\frac{1}{8}e^{4\cdot 2-8} + \frac{1}{2}\cdot 2$$

$$+ \frac{1}{8}e^{4\cdot 0-8} - \frac{1}{2}\cdot 0$$

$$= -\frac{1}{8} + 1 + \frac{1}{8}\cdot e^{-8}$$

$$= \frac{7}{8} + \frac{1}{8}e^{-8}$$

Megj: Ha téglalap lett volna T:

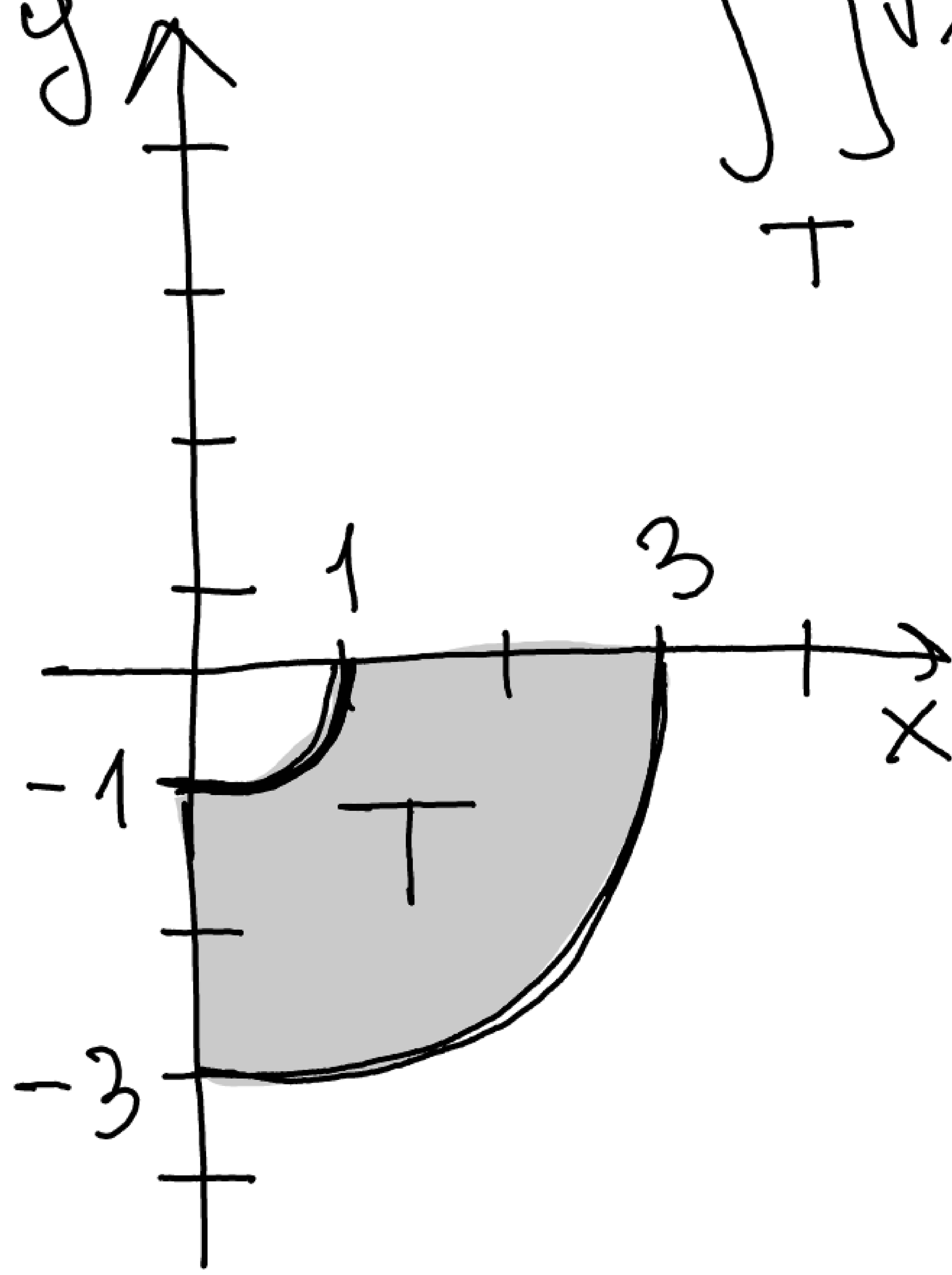


$$\iint_T e^{3x-2y}\,dA = \int_1^3\int_2^4 e^{3x-2y}\,dy\,dx$$

$$= \int_2^4\int_1^3 e^{3x-2y}\,dx\,dy$$

Megj.: Ha $f(x,y)=\sqrt{x^2+y^2}$

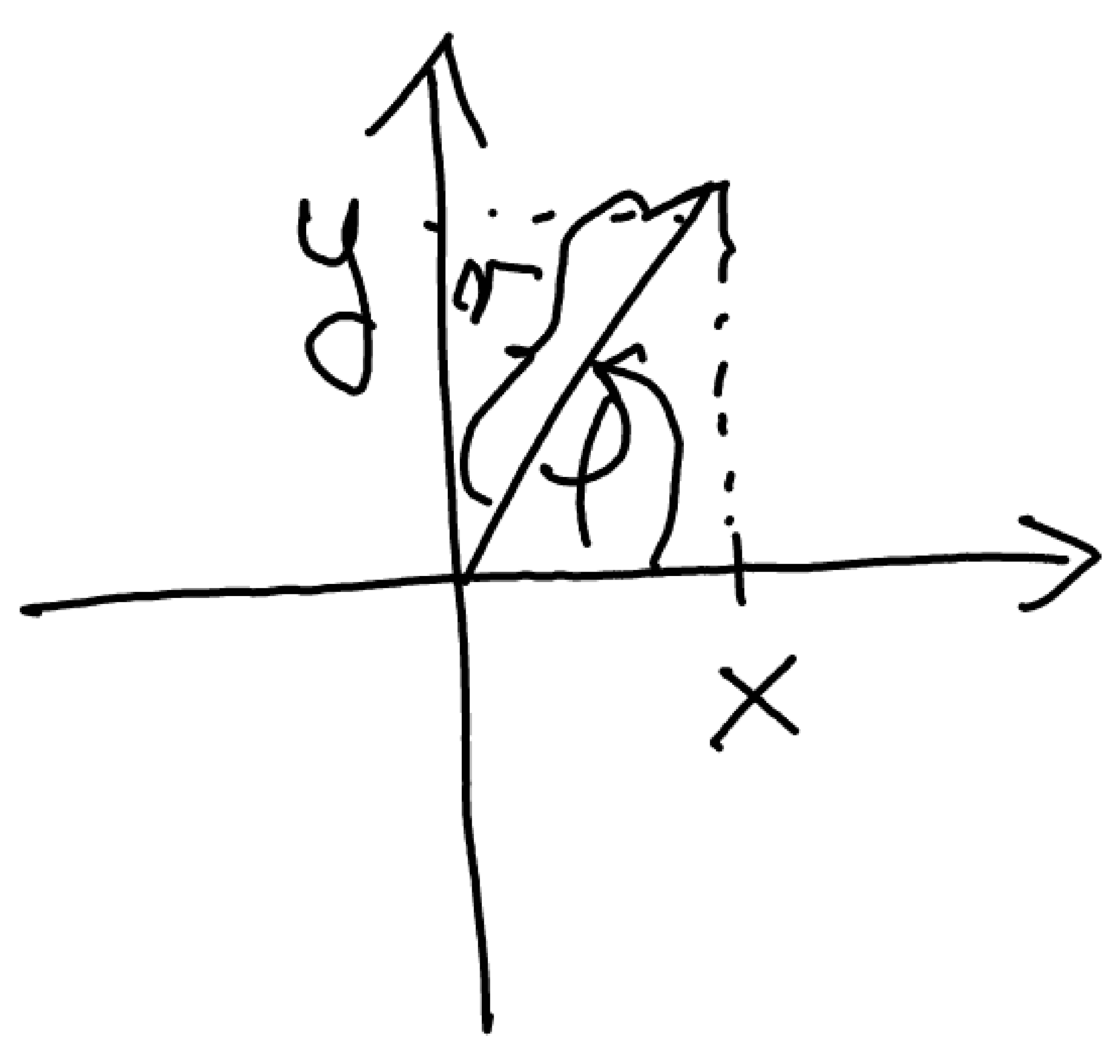
és T:



$$\iint_T \sqrt{x^2+y^2}\,dA = \int_1^3 \int_{\frac{3\pi}{2}}^{2\pi} \sqrt{r^2\cos^2(\varphi)+r^2\sin^2(\varphi)}\cdot r\,d\varphi\,dr$$

Polárkoordinátás helyettesítés:

$$\left.\begin{aligned} x &= r\cdot\cos(\varphi) \\ y &= r\cdot\sin(\varphi) \end{aligned}\right\} \quad \begin{aligned} r &\in [1,3] \\ \varphi &\in \left[\frac{3\pi}{2}, 2\pi\right] \end{aligned}$$



$$\int_{1}^{3}\int_{\frac{3\pi}{2}}^{2\pi}\sqrt{r^2\cos^2(\varphi)+r^2\sin^2(\varphi)}\cdot r\,d\varphi\,dr$$

$$=\int_{1}^{3}\int_{\frac{3\pi}{2}}^{2\pi}\sqrt{r^2}\cdot r\,d\varphi\,dr$$

$$=\int_{1}^{3}\int_{\frac{3\pi}{2}}^{2\pi}\underbrace{|r|}_{r}\cdot r\,d\varphi\,dr=\int_{1}^{3}\int_{\frac{3\pi}{2}}^{2\pi}r^2\,d\varphi\,dr$$