Probability Theory 2

Borel-Cantelli lemmas & Kolmogorov's SLLN 08.04.2025.

- **Borel-Cantelli lemmas**
  Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $A_n \in \mathcal{F}$ a sequence of events. Let us denote

$$B_\infty = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} A_n = \{A_n \text{ happen for infinitely many } n\}.$$

  - **1st lemma**

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty \quad \Rightarrow \quad \mathbb{P}(B_\infty) = 0$$

  - **2nd lemma**

$$(A_n)_{n=1}^{\infty} \text{ are fully independent} \quad \text{and} \quad \sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty \quad \Rightarrow \quad \mathbb{P}(B_\infty) = 1$$

- **Kolmogorov's inequality**
  Let $(X_k)_{k=1}^{\infty}$ be independent random variables with $\mathbb{E}(X_k) = 0$ and $\sigma_k^2 := \mathbb{E}(X_k^2) < \infty$. If we denote $S_j := X_1 + \cdots + X_j$, then for any $\lambda > 0$

$$\mathbb{P}\left(\max_{1 \le j \le n} |S_j| \ge \lambda\right) \le \frac{1}{\lambda^2} \sum_{k=1}^{n} \sigma_k^2.$$

- **Kolmogorov's two series theorem**
  Let $(X_k)_{k=1}^{\infty}$ be independent random variables with $\mathbb{E}(X_k^2) < \infty$. If

$$\lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{E}(X_k) \text{ exists}$$

  and

$$\sum_{k=1}^{\infty} \mathbb{D}^2(X_k) < \infty,$$

  then

$$\mathbb{P}\left(\lim_{n\to\infty} \sum_{k=1}^{n} X_k \text{ exists}\right) = 1$$

- **Kolmogorov's SLLN**
  Let $(X_k)_{k=1}^{\infty}$ be i.i.d. random variables with $\mathbb{E}(|X_k|) < \infty$. If we denote $\mu := \mathbb{E}(X_k)$ and $S_n := X_1 + \cdots + X_n$, then

$$\frac{S_n}{n} \xrightarrow{\textbf{a.s.}} \mu.$$