Probability Theory 2

Characteristic functions I and II 29.04.2025. and 06.05.2025.

- **Characteristic function**
  Let X be a real-valued random variable. The characteristic function of X is $\phi : \mathbb{R} \to \mathbb{C}$
  $$\phi(t) := \mathbb{E}\left(e^{itX}\right).$$

- **Some characteristic functions**

$\mathrm{BIN}(n,p)$	$t \mapsto \left(1 + \left(e^{it} - 1\right)\right)^n$
$\mathrm{POI}(\lambda)$	$t \mapsto \exp\left(\lambda\left(e^{it} - 1\right)\right)$
$\mathrm{GEO}(p)$	$t \mapsto \frac{p}{1-(1-p)e^{it}}$
$\mathrm{EXP}(\lambda)$	$t \mapsto \frac{\lambda}{\lambda - it}$
$\mathrm{UNI}(a,b)$	$t \mapsto \frac{e^{itb} - e^{ita}}{it(b-a)}$
$\mathrm{N}(\mu, \sigma^2)$	$t \mapsto \exp\left(it\mu - \frac{t^2\sigma^2}{2}\right)$
$\mathrm{CAU}(a,b)$	$t \mapsto \exp\left(ita - b\,\lvert t\rvert\right)$

- **Properties of the characteristic function**
  (i) $\phi(0) = 1$
  (ii) $|\phi(t)| \leq 1$
  (iii) $t \mapsto \phi(t)$ is uniformly continuous
  (iv) $t \mapsto \phi(t)$ is of positive type. I.e. for any $n \in \mathbb{N}^+$ and $z_1, \dots z_n \in \mathbb{C}$, $t_1, \dots, t_n \in \mathbb{R}$
  $$\sum_{k=1}^{n} \sum_{l=1}^{n} z_k \overline{z_l} \phi\left(t_k - t_l\right) \geq 0.$$

- **Bochner's theorem**
  A function $\phi : \mathbb{R} \to \mathbb{C}$ is a characteristic function if and only if the following properties hold
  (i) $\phi(0) = 1$
  (ii) $t \mapsto \phi(t)$ is continuous at $t = 0$
  (iii) $t \mapsto \phi(t)$ is of positive type

- **Moments**
  Supose for some $n \in \mathbb{N}$ $\mathbb{E}\left(|X^n|\right) < +\infty$. Then
  (i) $t \mapsto \phi(t)$ is C^n
  (ii) for any $0 \leq k \leq n$ $\phi^{(k)}(0) = i^k \mathbb{E}\left(X^k\right)$

- **Inversion formula**
  Let X be a random variable with cumulative distribution function F and characteristic function ϕ. If $a < b$ are continuity points of F, then

$$\mathbb{P}\left(a < X < b\right) = F(b) - F(a) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \phi(t)\, dt,$$

  and

$$\mathbb{P}\left(a < X < b\right) = F(b) - F(a) = \lim_{\sigma\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ita} - e^{-itb}}{it} e^{-\frac{\sigma^2 t^2}{2}} \phi(t)\, dt.$$

- **Inversion formula (in case of absolute continuity)**
  Let X be a random variable with probability density function f and characteristic function ϕ. If $f \in C^2$, then for any $x \in \mathbb{R}$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi(t)\, dt.$$

- **The characteristic function uniquely determines the distribution**
  Let X and Y be random variables with characteristic functions ϕ_X and ϕ_Y. Then the following are equivalent

  (i) $X \stackrel{d}{=} Y$

  (ii) for all $t \in \mathbb{R}$ $\quad \phi_X(t) = \phi_Y(t)$