Probability Theory 2

Concentration inequalities 18.03.2025.

- **Markov's inequality**
  Let $\mathbb{P}\left(X \geq 0\right) = 1$. Then for any $\lambda > 0$
  $$\mathbb{P}\left(X \geq \lambda\right) \leq \frac{\mathbb{E}\left(X\right)}{\lambda}.$$

- **Chebyshev's inequality**
  Let $\mathbb{E}\left(|X|\right) < \infty$. Then for any $\lambda > 0$
  $$\mathbb{P}\left(|X - \mathbb{E}\left(X\right)| \geq \lambda\right) \leq \frac{\mathbb{D}^2\left(X\right)}{\lambda^2}.$$

- **Chernoff-Rubin Bound**
  Let $X_1, X_2, \ldots, X_n$ be independent random variables. Suppose $\mathbb{P}\left(|X_i - \mathbb{E}\left(X_i\right)| \leq K\right) = 1$. Let us denote
  $$S_n = X_1 + X_2 + \cdots + X_n.$$
  Then for any $0 \leq \varepsilon \leq 1$ and $0 \leq \lambda \leq \varepsilon \frac{\mathbb{D}(S_n)}{K}$
  $$\mathbb{P}\left(S_n - \mathbb{E}\left(S_n\right) \geq \lambda \mathbb{D}\left(S_n\right)\right) \leq \exp\left(-\frac{\lambda^2}{2\left(1+\varepsilon\right)}\right)$$
  and
  $$\mathbb{P}\left(|S_n - \mathbb{E}\left(S_n\right)| \geq \lambda \mathbb{D}\left(S_n\right)\right) \leq 2\exp\left(-\frac{\lambda^2}{2\left(1+\varepsilon\right)}\right).$$

- **Bernstein's inequality**
  Let $X_1, X_2, \ldots, X_n$ be independent random variables. Suppose $\mathbb{P}\left(|X_i - \mathbb{E}\left(X_i\right)| \leq K\right) = 1$. Let us denote
  $$S_n = X_1 + X_2 + \cdots + X_n.$$
  Then for any $0 \leq \lambda \leq \frac{\mathbb{D}(S_n)}{K}$
  $$\mathbb{P}\left(S_n - \mathbb{E}\left(S_n\right) \geq \lambda \mathbb{D}\left(S_n\right)\right) \leq \exp\left(-\frac{\lambda^2}{2\left(1+\lambda\frac{K}{\mathbb{D}(S_n)}\right)^2}\right)$$
  and
  $$\mathbb{P}\left(|S_n - \mathbb{E}\left(S_n\right)| \geq \lambda \mathbb{D}\left(S_n\right)\right) \leq 2\exp\left(-\frac{\lambda^2}{2\left(1+\lambda\frac{K}{\mathbb{D}(S_n)}\right)^2}\right).$$

- **Hoeffding's inequality**
  Let $X_1, X_2, \ldots, X_n$ random variables such that $\mathbb{P}\left(a_i \leq X_i \leq b_i\right) = 1$. Let us denote

$$S_n = X_1 + X_2 + \cdots + X_n.$$

  Then for any $\lambda > 0$

$$\mathbb{P}\left(S_n - \mathbb{E}\left(S_n\right) \geq \lambda\right) \leq \exp\left(-\frac{2\lambda^2}{\sum_{i=1}^{n}\left(b_i - a_i\right)^2}\right)$$

  and

$$\mathbb{P}\left(\left|S_n - \mathbb{E}\left(S_n\right)\right| \geq \lambda\right) \leq 2\exp\left(-\frac{2\lambda^2}{\sum_{i=1}^{n}\left(b_i - a_i\right)^2}\right).$$