Probability Theory 2

Method of characteristic functions — 20.05.2025.

- **Lévy's continuity theorem**
  Let X_n be random variables and ϕ_n their characteristic functions. If

  (i) for any $t \in \mathbb{R}$ $\lim\limits_{n\to\infty} \phi_n(t) = \phi(t)$ exists

  (ii) $t \mapsto \phi(t)$ is continuous at $t = 0$

  then
  $$X_n \xrightarrow{d} X,$$
  where ϕ is the characteristic function of X.

- **Cramer-Slutsky theorem**
  Let X_n, Y_n, Z_n and X be random variables over the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. If

  (i) $X_n \xrightarrow{d} X$

  (ii) $Y_n \xrightarrow{\mathbb{P}} y \in \mathbb{R}$

  (iii) $Z_n \xrightarrow{\mathbb{P}} z \in \mathbb{R}$

  then
  $$X_n Y_n + Z_n \xrightarrow{d} X \cdot y + z.$$

- **Central limit theorem**
  Let X_n be i.i.d. random variables with $\mu := \mathbb{E}\left(X_n\right)$ and $\sigma^2 := \mathbb{D}^2\left(X_n\right) < \infty$. Let
  $$S_n = X_1 + \cdots + X_n.$$
  Then
  $$\frac{S_n - n\mu}{\sqrt{n}\sigma} \xrightarrow{d} X,$$
  where $X \sim \mathrm{N}\left(0, 1\right)$.

- **Some characteristic functions**

BIN(n,p)	$t \mapsto \left(1 + p\left(e^{it} - 1\right)\right)^n$
POI(λ)	$t \mapsto \exp\left(\lambda\left(e^{it} - 1\right)\right)$
GEO(p)	$t \mapsto \frac{p}{1-(1-p)e^{it}}$
EXP(λ)	$t \mapsto \frac{\lambda}{\lambda - it}$
UNI(a,b)	$t \mapsto \frac{e^{itb}-e^{ita}}{it(b-a)}$
N(μ, σ^2)	$t \mapsto \exp\left(it\mu - \frac{t^2\sigma^2}{2}\right)$
CAU(a,b)	$t \mapsto \exp\left(ita - b\left\lvert t\right\rvert\right)$