Probability Theory 2

2nd Exercise Sheet: Convolutions II 18.02.2025.

2.1 We say that a random variable X has Cauchy distribution with parameters $m \in \mathbb{R}$ and $\tau > 0$ (notation: $\mathrm{CAU}(m, \tau)$) if its density function is

$$f_{m,\tau}(x) = \frac{\tau}{\pi(\tau^2 + (x-m)^2)}$$

for $x \in \mathbb{R}$. Show that for any $m_1, m_2 \in \mathbb{R}$ and $\tau_1, \tau_2 > 0$ we have $\left(f_{m_1,\tau_1} * f_{m_2,\tau_2}\right)(x) = f_{m_1+m_2,\tau_1+\tau_2}(x)$.

HW 2.2 Let $X, Y > 0$ be positive and independent random variables with distribution functions F and G respectively. Give the distribution of XY!

2.3 Let X and Y be independent random variables such that $X \sim UNI[0,1]$ and Y has k-times continuously differentiable distribution function $F(y) = \mathbb{P}(Y < y)$, where $k \in \{0, 1, \ldots\}$. Show that the distribution of $X + Y$ is $(k+1)$-times continuously differentiable.

2.4 Let $X_1, X_2, \ldots, X_n, \ldots$ be i.i.d random variables with common distribution $\mathbb{P}(X_i = 0) = \frac{1}{2} = \mathbb{P}(X_i = 1)$. Let $Y := \sum_{n=1}^{\infty} 2^{-n} X_n$. (The sum is convergent with probability 1!) Prove that the distribution of Y is uniform on the interval $[0, 1]$.

HW* 2.5 Let $X_1, X_2, \ldots, X_n, \ldots$ be i.i.d random variables with common distribution $UNI(0, 1)$. Let $Y := \sum_{n=1}^{\infty} 2^{-n} X_n$. (The sum is convergent with probability 1!) Prove that the distribution function $F(y) := \mathbb{P}(Y < y)$ of Y is continuous. Moreover, show that F is infintely differentiable but nowhere analytic. (*Hint:* For the last part show that the radius of convergence of the Taylor series is zero for every point in [0,1].)

2.6 Prove that if X and Y are i.i.d. standard normal random variables, and a and b real numbers then $U = aX + bY$ és $V = bX - aY$ are also independent. What distribution do U and V have?

2.7 For a given $\lambda, \nu > 0$, denote $\mathrm{GAM}(\nu, \lambda)$ the distribution, of which density function is

$$f_{\nu,\lambda}(x) := \frac{1}{\Gamma(\nu)} \lambda^{\nu} x^{\nu-1} e^{-\lambda x} \mathbb{1}_{\{x>0\}}.$$

Calculate the value of $B(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1} dx$ for every fixed parameters $a, b > 0$. (*Hint:* Use the definition of $\Gamma(a)\Gamma(b)$ and the substitution $x = zt, y = z(1-t)$ to calculate the double integral $\int_0^{\infty} \int_0^{\infty} x^{a-1} e^{-x} y^{b-1} e^{-y}\, dx dy$ in two ways!)

HW 2.8 Calculate the moments of the distribution $\mathrm{GAM}(\nu, \lambda)$.

2.9 For every $a, b > 0$, denote $BETA(a, b)$ the distribution, of which density function is

$$f_{a,b}(x) := \frac{1}{B(a,b)} x^{a-1} (1-x)^{b-1} \mathbb{1}_{\{x \in [0,1]\}}.$$

Calculate the moments of $BETA(a, b)$! (The constant $B(a, b)$ is defined in **2.7**.)

HW 2.10 Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with standard normal distribution. Show that $\sum_{i=1}^{n} X_i^2 \sim \mathrm{GAM}\left(\frac{n}{2}, \frac{1}{2}\right)$.