Probability Theory 2

3rd Exercise Sheet: Generating Function I 25.02.2025.

3.1 Determine the probability generating function $P(z) = \mathbb{E}(z^X)$ of $BIN(n,p)$, $POI(\lambda)$ and $GEO(p)$.

3.2 Can the following functions be a prob. generating function of some distribution?

$$(a) \quad \exp\left(\frac{z-1}{\lambda}\right), \quad \lambda > 0; \qquad (b) \quad \frac{(z+1)^6}{64}; \quad (c) \quad \frac{2}{2-z}; \qquad (d) \quad \frac{2}{1+z}.$$

3.3 Let Λ be a random variable with distribution $EXP(\mu)$ and let X be the random variable, which conditional distribution is $POI(\Lambda)$ conditioned on Λ. Determine the distribution of X.

HW **3.4** Let U be a random variable with distribution $UNI(0,1)$, and let X be the random variable, which conditional distribution is $BIN(n,U)$ conditioned on U. Determine the distribution of X.

3.5 Let X be an $\mathbb{N}$ valued random variable and denote $P(z)$ the prob. generating function of X. Give the prob. gen. function of $Y := X+1$ and $Z := 2X$.

3.6 Let X be an $\mathbb{N}$ valued random variable and denote $P(z)$ the prob. generating function of X. Give the generating functions of the following sequences: $a_n := \mathbb{P}(X \leq n)$, $b_n := \mathbb{P}(X < n)$, $c_n := \mathbb{P}(X \geq n)$, $d_n := \mathbb{P}(X > n+1)$ and $e_n := \mathbb{P}(X = 2n)$. (Note that these are not prob. distributions.)

3.7 Let $\xi_1, \xi_2, \dots$ be i.i.d random variables with distribution: $\mathbb{P}\big(\xi_i = 1\big) = p$, $\mathbb{P}\big(\xi_i = 0\big) = 1-p$, where $0 < p < 1$. Let

$$\nu_{\alpha\beta} := \min\{n \geq 2 : \xi_{n-1} = \alpha,\ \xi_n = \beta\}, \qquad \alpha, \beta \in \{0,1\}.$$

Determine the prob. gen. function of $\nu_{\alpha\beta}$, and using this, the expected value and variance, for every possible combination of α, β.

HW **3.8** Let $\xi_1, \xi_2, \dots$ be i.i.d. $\mathbb{N}$-valued random variables such that $\mathbb{P}(\xi_j = n) = p_n$ for $n = 0, 1, 2, \dots$. Let $\nu = \min\{k \geq 2 : \xi_{k-1} = \xi_k\}$. Find the probability generating function and the expected value of ν!

(Hint: express $\mathbb{E}(z^\nu | \xi_1 = n)$ by using the second toss with $\mathbb{E}(z^\nu)$.)

3.9 We call the distribution of a random variable X *infinitely divisible* for every $n \geq 1$ there exist i.i.d random variables $Y_1^n, \dots, Y_n^n$ such that $\sum_{k=1}^n Y_k^n$ and X have the same distribution. Are the binomial, Poission and geometric distributions infinitely divisible?

HW* **3.10** Let $A_0 A_1 \dots A_n$ be an $(n+1)$-sided convex polygon on the plane. Let $a_1 = 1$, and for $n \geq 2$ denote a_n the number possible decompositions of this polygon to $n-1$ many triangles, by using $(n-2)$-many (pairwise) not intersecting diagonal. Prove the identity for $n \geq 2$

$$a_n = a_1 a_{n-1} + a_2 a_{n-2} + \dots + a_{n-1} a_1 = \sum_{k=1}^{n-1} a_k a_{n-k}.$$

Using the identity above, calculate the generating function of a_n, using that give an explicit formula for a_n.

3.11 Let $X_1, X_2, \dots$ be i.i.d. $\mathbb{N}$-valued random variables and let ν be an $\mathbb{N}$-valued random variable independent of X_i's. Let $Y = \sum_{k=1}^{\nu} X_k$. Show that

$$\mathbb{E}\,(Y) = \mathbb{E}\,(\nu)\,\mathbb{E}\,(X_1) \qquad \text{and} \qquad \mathbb{D}^2(Y) = \mathbb{D}^2(\nu)\mathbb{E}(X_1)^2 + \mathbb{E}(\nu)\mathbb{D}^2(X_1).$$

HW 3.12 Let $X_1, X_2, \dots$ be i.i.d random variables with distribution (optimistic) $GEO(p_1)$ ($\mathbb{P}(X_i = k) = p_1(1-p_1)^{k-1}$, $k = 1, 2, \dots$), and let ν be a random variable independent of X_i's and with distribution (optimistic) $GEO(p_2)$ ($\mathbb{P}(\nu = k) = p_2(1-p_2)^{k-1}$, $k = 1, 2, \dots$). Show by using prob. gen. functions that

$$\sum_{i=1}^{\nu} X_i \text{ has distribution (optimistic) } GEO(p_1 p_2)!$$

3.13 Show that for every $0 < p < 1$ there exists a prob. sequence $p_0, p_1, \dots$ (i.e. $p_k \geq 0$ and $\sum_{k=0}^{\infty} p_k = 1$) and a parameter $\lambda > 0$, such that $\sum_{i=1}^{\nu} X_i$ has distribution $GEO(p)$, where $X_1, X_2, \dots$ are i.i.d random variables with distribution $\mathbb{P}(X_i = k) = p_k$ for every $k \geq 1$, and ν is independent of X_i with distribution $POI(\lambda)$.