

Probability Theory 2

4th Exercise Sheet: Generating Function II

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- 4.1** Let $G(s, t)$ be the joint probability generating function of the random variables X and Y . That is $G(s, t) = \mathbb{E}(s^X t^Y)$. Prove that $G(s, 1)$ is the prob. gen. function of X , and $G(1, t)$ is the prob. gen. function of Y . Moreover, show that

$$\mathbb{E}(XY) = \frac{\partial^2}{\partial s \partial t} G(s, t) \Big|_{s=t=1}.$$

What does the relation $G(s, t) = G(s, 1) \cdot G(1, t)$ mean? What is $G(s, s)$?

- HW 4.2** Prove that the function

$$G(x, y, z, w) = \frac{1}{8} (xyzw + xy + yz + zw + zx + yw + xw + 1)$$

is the joint prob. generating function of 4 random variables such that any two and any three out of them are independent, but all the four are not independent.

- HW 4.3** Let us consider a branching process for which the expected value of the offsprings is ν and the variance is σ^2 . Denote X_n the number of individuals in the n th generation. That is, $X_0 = 1$, X_1 is the number of children of the first individual, etc. Find the variance of the n th generation!

- (a) Give a formula for $\mathbb{D}^2(X_n)$ in case $\nu \neq 1$!
- (b) Give a formula for $\mathbb{D}^2(X_n)$ in case $\nu = 1$!

(Hint: The formula appeared in the lecture. Prove by induction!)

- 4.4** Denote $\theta(p)$ the probability that a branching process with offspring distribution $GEO(p)$ never extinct. Plot the graph of $p \mapsto \theta(p)$!

- 4.5** Let us consider a branching process, for which the probability generating function of the offsprings is $P(z)$. Denote X the size of the whole population (i.e. the number of all individuals who ever lived). Denote $Q(z) = \mathbb{E}(z^X)$. Prove that $Q(z)$ is the inverse of $z/P(z)$!

- 4.6** (a) What is the probability that a branching process with successor distribution $GEO(1/2)$ survives until the n th generation? (Here, Y has distribution $GEO(1/2)$ if $\mathbb{P}(Y = k) = (1/2)^{k+1}$ for every $k \geq 0$.)
(b) Denote X the size of the complete population. That is, if Y_n denotes the cardinality of the n th generation then $X = \sum_{n=0}^{\infty} Y_n$. What is

$$\lim_{k \rightarrow \infty} k^{3/2} \mathbb{P}(X = k) = ?$$

- 4.7** Let $\xi_1, \xi_2, \dots$ be i.i.d random variables such that $\mathbb{P}(\xi_i = \pm 1) = \frac{1}{2}$. Denote $S_n = \sum_{i=1}^n \xi_i$ the simple random walk on $\mathbb{Z}$. Let $\tau = \min\{n \geq 1 : S_n = 1\}$. Calculate $\mathbb{P}(\tau = k)$! What is the limit

$$\lim_{k \rightarrow \infty} k^{3/2} \mathbb{P}(\tau = 2k - 1) = ?$$

(Hint: We saw that the prob. gen. function of τ is $P(z) = \frac{1 - \sqrt{1 - z^2}}{z}$ and use that $\binom{1/2}{k} = \frac{(2k)}{k} \frac{(-1)^{k+1}}{2^{2k}(2k-1)}.$)

- HW 4.8** An amoebae can do during a day only two things: it splits into two with probability $\frac{1}{2}$, and dies with prob. $\frac{1}{2}$. Denote X the number of the branches of the branching process. Show that X and τ have the same distribution, where τ is defined in **4.7**. What is the meaning of this?

HW* 4.9 Let $\xi_1, \xi_2, \dots$ be i.i.d. random variables such that $\mathbb{P}(\xi_i = 1) = p$ and $\mathbb{P}(\xi_i = -1) = q$ with $p + q = 1$ and $q \neq p$. Denote by $S_n = \sum_{k=1}^n \xi_k$ the asymmetric random walk on $\mathbb{Z}$. Let $\nu = \#\{n \geq 1 : S_n = 0\}$ be the number of returns to the origin and let $\lambda = \sup\{n \geq 1 : S_n = S_0 = 0\}$ be the last return time. Find the prob. gen. function of ν and λ !

(*Hint:* We have seen that the prob. gen. of the first return time $\rho = \min\{n \geq 1 : S_n = 0\}$ is $G(z) = 1 - \sqrt{1 - 4pqz^2}$. Express $\mathbb{P}(\nu = k)$ by using $\mathbb{P}(\rho < \infty)$ and λ by using ν and ρ !)

4.10 Consider the (infinite) graph $\mathbb{G}_g$, which is a homogeneous tree with degree g and a symmetric random walk on it. That is, let S_n be a random walk on $\mathbb{G}_g$, which starts at a given vertex (the root) and in every time unit, it chooses one out of the g neighbours with uniform probability g^{-1} and steps there. Determine the functions Φ , F and L . ($\Phi(z)$: the prob. gen. function of the first hitting time of a given neighbour, $F(z)$ the prob. gen. function of the first return to the root; $L(z)$ the prob. gen. function of the last hitting time of the root.)